Managing the Quality of Public Goods
with Stock and Flow Controls

Nathaniel O. Keohane  Benjamin Van Roy  Richard J. Zeckhauser†
Yale University  Stanford University  Harvard University

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Abstract

We consider a class of problems, which we call “SFQ” problems, in which both stocks and flows can be controlled to promote the quality of a public good, such as environmental quality or public infrastructure. The analysis treats quality as a stock. Under the optimal policy, periodic restoration of the stock complements positive but variable abatement of the flow. When deterioration is more rapid or highly variable, or when abatement is more expensive relative to restoration, the optimal policy relies relatively more on restoration.

In many cases of interest – e.g., a municipal dump or a highway – quality deteriorates at a rate determined by the actions of myriad firms or individuals. A state-dependent flow tax, equal to the present value of marginal damages, would provide efficient incentives for abatement. This tax rises at first as quality worsens, but eventually falls as restoration nears. Moreover, the optimal tax rate may be lower when the underlying deterioration is more rapid. In concert with a tax on variance, this flow tax would raise precisely the revenue required to pay for restoration.

We discuss the implications of the SFQ model for a range of real-world problems in the environmental arena, and for the management of public infrastructure. But the lessons are general, and we briefly discuss how they apply to private stocks of physical and human capital.

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†Corresponding author. Mailing address: Richard Zeckhauser, John F. Kennedy School of Government, 79 John F. Kennedy St., Cambridge, MA 02138; email richard.zeckhauser@harvard.edu; telephone (617) 495-1174; fax (617) 384-9340.
1 Introduction

Public policies across a broad range of issues involve maintaining the quality of a valued resource stock. Thus, we try to prevent the deterioration of environmental quality, control the illicit use of drugs, and keep our roads well paved. The economist’s typical policy prescription in such settings is to identify the level where the marginal cost of maintaining quality equals its marginal benefit, and then to stay there. For example, pollution regulations may specify a target level of emissions (in the case of local air pollution), or a target concentration of carbon dioxide (in the case of global climate change). Such an approach is optimal when quality can be maintained only by slowing the rate of deterioration, e.g., through pollution abatement.

However, there are often economies of scale in moving a stock to a desired level of quality. In such cases, periodic restoration of the stock will be worthwhile. Though restoration and abatement policies have been studied separately in relation to a range of public policy issues, we are unaware of any studies that bring them together. This paper studies the joint role of abatement and restoration in managing public capital stocks, using environmental quality and public infrastructure as applications.

A simple example illustrates our theory. Consider the Massachusetts Avenue bridge that is the main link between Cambridge and Boston. The rate at which it deteriorates can be slowed by regular maintenance, or by traffic restrictions such as weight limits. Eventually, however, it will wear out (a stage that is near) and have to be replaced with a new bridge. Over the course of its lifetime, optimal maintenance for the bridge varies. Early on, maintenance increases with time, as it requires increasing attention. Toward the end of its life, however, optimal maintenance declines, since the future benefits of better quality for the current bridge decline. There is no need to fill the cracks in a bridge that is about to be replaced.

The motivating observation of this paper is that the simple logic of the maintenance and replacement of a bridge applies to the management of the quality of a wide range of public goods. In some instances, e.g., the state of a bridge or highway, quality will be just that; in others, e.g., illicit drug use, quality should be thought of as the quantity of some good (or bad) activity or stock. A canonical example in the environmental arena is the accumulation and treatment of waste at landfills or generating sites. Optimal management both slows the generation of new wastes and periodically cleans up accumulated stocks. Similar dynamics play out in the management of a variety of other public goods: a reservoir fills with sediment until the dam must be replaced; a
groundwater aquifer is drawn down and eventually recharged; potholes are patched until a street is
dug up and repaved; drug dealing is kept under control through a police presence until a massive
drug sweep is undertaken.

In each of these settings, two distinct approaches are available to manage the quality of the
public good: boosting the quality of the public good, and slowing the rate at which it deteriorates.
Hence both stocks and flows can be controlled to promote quality. We refer to this class of problems
as “SFQ” problems. In this paper, we develop a general model of the optimal management of a
resource stock when flows are controllable and restoration of the stock is feasible.

To distinguish formally between the two management strategies, we assume that the costs of flow
control (which we call abatement) increase on the margin, but that stock control (or restoration)
exhibits economies of scale, so that discrete improvements are potentially desirable. Such scale
economies are likely to obtain in many settings. For example, cleaning up a hazardous waste site
may require hauling the soil away for off-site incineration, in which case the costs vary little with
the concentration of the contaminant in the soil. Similarly, there are high fixed costs involved
in capping a landfill or paving city streets. In many settings, the source of the nonconvexity is
institutional rather than technological. In fiscal policy, for example, even small tax increases may
face substantial political opposition, making the political costs of change largely fixed. Thus, tax
reform is a rare event. As will become apparent below, what is crucial to our analysis is that there
are economies of scale “at the bottom” – that is, that the costs of restoration do not increase too
rapidly as the quality of the stock diminishes.

Consider the environmental case. Given economies of scale in restoration, the optimal policy
calls for restoring the resource whenever quality falls to a sufficiently low level. At states above that
point, the new flow is abated at a rate that varies with the current quality of the resource. After
restoration occurs, deterioration resumes, quality starts to decline (albeit stochastically), and the
cycle repeats. The optimal trade-off between abatement and restoration depends on the magnitude
and variability of flows, the relative costs of the two strategies, and the discount rate. If flows are low
enough, or if abatement is sufficiently inexpensive relative to restoration, optimal abatement may
be sufficiently intense to offset the expected deterioration, producing an equilibrium in expectation.
Even in this case, restoration remains an option if unexpected shocks reduce the stock of quality
sufficiently; hence its availability influences the optimal abatement path. When deterioration is
more rapid or more variable, or when restoration is relatively less costly, the optimal policy relies
more on restoration.
In many settings, the stock deteriorates at a speed determined by the actions of myriad firms or individuals. This raises the question of policy design. In particular, we consider the natural case where abatement must be undertaken by a large number of firms but where restoration is implemented by the center. A landfill is a natural example. Private parties (perhaps influenced by taxes or other policy measures) control the flow into it. The government ultimately caps it and restores the site.

The center now has two tasks: first, it must align private incentives for abatement with social welfare; second, it must raise revenue to cover the costs of restoration. The optimal abatement path is achieved by charging a time-varying flow tax equal to the present value of marginal damages. The tax has a surprising pattern. As the quality of the stock decreases, this optimal tax rises at first, but it eventually falls as the state worsens and restoration nears. Moreover, the optimal tax rate may be lower when there is more pressure on quality (e.g., unregulated waste flow, or cars and trucks on the highway). However, the revenue from such a flow tax will not in general equal the cost of restoration. We show that the difference could be collected in the form of a tax on variance, with the intuitively appealing property that such a tax would penalize firms for introducing variance when doing so reduced expected welfare, and would reward them for variance that was welfare-improving. In the deterministic case, no variance tax would be needed, and the revenues from the flow would precisely equal the cost of restoration.

The model of this paper melds two instruments that have typically been considered in isolation. Conventional models of the optimal management of stock pollutants have modeled abatement alone.\(^1\) The optimal policy in that setting equates the marginal benefit of reducing pollution, adjusted for the discount rate and the decay rate of the stock, to the marginal cost of abating it. A steady state is reached in which optimal abatement efforts just keep up with net new accumulation (Falk and Mendelsohn 1993; Keeler, Spence, and Zeckhauser 1971; Plourde 1972; Plourde and Yeung 1989; Smith 1972).\(^2\) During the transition to the steady state, the shadow value of environmental quality rises steadily, and the optimal tax rises with it (Farzin 1996). In contrast, when restoration offering economies of scale is available, the optimal policy may entail periodic

\(^1\)In our model, of course, the “stock” is a good (resource quality) rather than a bad (pollution); but that is only a difference in sign. The important distinction is that we consider controls on both stocks and flows.

\(^2\)A few models of optimal cleanup of an accumulated stock of pollution have considered restoration but not abatement. Caputo and Wilen (1995) assume that cleanup costs are convex. As a result, the optimal solution stops short of complete cleanup (they let natural degradation finish the process), as long as when pollution approaches zero so does its marginal damage. Phillips and Zeckhauser (1998) assume economies of scale in cleanup, but consider the problem in a static setting and hence ignore abatement.
restorations punctuating long periods of deterioration that is only partially offset by abatement.

In the same way, endogenous maintenance has significant consequences for optimal replacement of physical capital. In the theoretical literature on capital investment, attention has centered on replacement rather than maintenance— that is, on restoration rather than abatement. Optimal investment in these models typically follows an \((S,s)\) policy (Arrow, Harris, and Marschak 1951). We show how introducing abatement changes the “trigger level” at which restoration is optimal.

The next section introduces the basic model, and formally defines our notions of abatement and restoration. Section 3 develops the theoretical results. Second 4 considers implementation of the optimal policy in a decentralized setting. Section 5 illustrates these results with examples of real-world SFQ problems, highlighting several environmental applications as well as public infrastructure. Section 6 concludes.

2 Model framework

Our model considers the management of a valued resource with a quality level that changes over time. In the case of accumulating waste, for example, quality might be measured by the volume of waste: the smaller the amount, the higher the level of environmental quality. We denote quality at time \(t\) by a real number \(x_t\), with larger values of \(x_t\) representing more desirable states. (Note that we use “quality” to denote the state of the resource. How the quality of a resource is valued will be captured in the utility function.) We normalize the initial quality level to be equal to zero, so that \(x_0 = 0\), and we shall be working mostly with negative values for \(x\). To keep things simple, we assume for the time being that there is a “manager” of the resource, who implements abatement and restoration policies in order to maximize the expected net present value of social welfare.

We model the deterioration of the resource, absent the efforts of the resource manager, as a random variable with drift. For the case of an environmental resource, for example, ongoing damage to the resource (new pollution) is partially offset by natural recovery (decay of accumulated pollutants); our specification captures both effects. In particular, cumulative deterioration up to time \(t\), denoted by \(z_t\), is assumed to follow a Brownian motion with drift rate \(\mu > 0\), variance

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3 For models of investment in physical capital, see Feldstein and Rothschild (1974) and Abel and Eberly (1994). Nickell (1975) considers maintenance as well as replacement, but he models maintenance as exogenously determined. For a model of optimal consumption of durable goods, see Grossman and Laroque (1990).

4 The manager could be the administrator of a regulatory agency that issues rules or provides rewards to influence the behavior of private-sector firms. We focus here on the behavior that a central planner would prescribe, and defer issues of instrument design to the discussion in section 4 below.
rate \( \sigma^2 \), and \( z_0 = 0 \). Hence, deterioration evolves according to \( z_t = \mu t - \sigma w_t \), where \( w_t \) follows a standard Brownian motion. Unless the manager curbs the rate of deterioration or restores the resource, therefore, quality at time \( t \) is given by

\[
x_t = -\mu t + \sigma w_t.
\]  

(1)

Intuitively, \( \mu \) can be thought of as the “average” rate of deterioration of the resource: for example, average pollution emissions minus natural decay. Throughout the analysis, we will refer to \( \mu \) as the “flow rate,” and will use the terms “flow” and “deterioration” interchangeably. The random term in equation (1) captures random variations in the processes of damage and natural recovery. The relative values of the flow rate (\( \mu \)) and the variance rate (\( \sigma^2 \)) will vary with the setting. In contexts where natural recovery is negligible – e.g., a bridge – \( \sigma^2 \) will be small relative to \( \mu \).

### 2.1 Utility and cost functions

We assume that society’s benefit from the resource at any point in time depends only on the level of quality. Thus, at time \( t \) society derives a flow of utility \( u(x_t) \) from the availability of the resource.\(^5\) The social rate of time preference is denoted by \( \alpha > 0 \). We further assume that the utility function has the following properties.

**Assumption 1** The utility function \( u \) is twice continuously differentiable, with \( u < 0, u' > 0, u'' < 0 \), and \( u' \) unbounded above. Furthermore, \( E_x \left[ \int_0^\infty e^{-\alpha t} u(x_t) dt \right] \) is finite for all \( x \), where \( E_x \) denotes the expectation conditional on an initial state \( x \).

Note that utility takes negative values; the utility function can be thought of as the negative of a convex loss function.\(^6\)

We define abatement as a reduction in the rate of deterioration: abating at rate \( a \) slows the expected deterioration rate from \( \mu \) to \( \mu - a \). Crucially, its costs are increasing on the margin.

**Assumption 2** The abatement cost function \( c : [0, \infty) \) is twice continuously differentiable with \( c \geq 0, c(0) = 0, \) and \( c'' \geq \epsilon \) for some \( \epsilon > 0 \).

\(^5\)We ignore issues such as population growth or changes in income, which could make the utility function time-dependent. With a growing population, for example, one might scale the utility function to the size of the population. If abatement costs remained constant, the optimal level of abatement at a given level of quality would increase over time. On the other hand, we might expect that abatement costs and the drift rate \( \mu \) might be greater for a larger population.

\(^6\)The assumption of negative utility is made for convenience. A reader uncomfortable with negative utility may add any constant term she wishes to make utility positive over its relevant range, without affecting the results.
We assume that a finite maximum feasible rate of abatement exists, denoted \( \bar{a} \).\(^7\) This ceiling may be higher than the mean flow rate \( \mu \). Hence our model encompasses (but does not impose) the possibility that abatement may more than fully offset deterioration. In such a case, “abatement” results in a positive rate of change in quality – but with increasing marginal costs.

Restoration corresponds to an improvement in quality – affecting the stock directly, rather than by slowing deterioration. We make two simplifying assumptions to ease exposition; neither are crucial to our results. First, we assume that the manager can restore the resource from any state \( x_t \) to a certain high level, which we normalize as \( x = 0 \). (We relax this assumption of a fixed destination in section 3.4.1, below.) Second, we assume an extreme form of nonconvexity: a positive fixed cost of restoration to \( x = 0 \), with zero marginal cost to starting at a lower point.

**Assumption 3** The cost of restoring quality from any state \( x_t \) to \( x = 0 \) is independent of \( x_t \) and is given by \( C > 0 \).

Thus the cost of restoration is “destination-driven” in the sense of Phillips and Zeckhauser (1998): it depends on the ultimate level of quality, rather than the initial level (or the size of the quality gain). While this assumption simplifies the model, our results hold for cost functions exhibiting less extreme economies of scale, as we discuss below in section 3.4.2.

### 2.2 Abatement and restoration policies

An **abatement policy** \( a(x) \) specifies the abatement level as a function of the state \( x \).\(^8\) Under a **restoration policy** \( R \), restoration occurs whenever the state \( x_t \) lies in the set \( R \).\(^9\) Given a combined abatement–restoration policy \( (a, R) \), the state of the resource evolves according to

\[
x_t = \int_{s=0}^{t} (a(x_s) - \mu) ds + \sigma w_t - \sum_{i | T_i < t} x_{T_i},
\]

where \( T_i \) is the time at which the \( i \)th restoration occurs. Hence starting from an initial state \( x \), the infinite-horizon expected discounted utility can be written as

\[
E_{x}^{a,R} \left[ \sum_{t=0}^{\infty} e^{-\alpha t} (u(x_t) - c(a(x_t))) dt - \sum_{i=1}^{\infty} e^{-\alpha T_i} C \right].
\]

\(^7\)The assumption of a ceiling on abatement provides a measure of generality. In some cases of interest, the manager may have limited abilities to stem or particularly to reverse the flow of deterioration. This assumption is completely innocuous: the ceiling can always be set high enough that the probability it binds is vanishingly small.. Moreover, an optimal abatement policy can still be shown to exist even if we allow abatement to be unbounded.

\(^8\)More formally, an abatement policy is a mapping \( a : \mathbb{R} \mapsto [0, \bar{a}] \). We shall restrict our attention to optimal stationary policies – policies that depend solely on the state \( x \). An alternative approach would make the manager’s problem one of choosing an optimal stochastic process \( \{a_t\} \) measurable with respect to the filtration generated by \( \{w_t\} \). However, one can show that such an optimal process can be produced by letting \( a_t = a(x_t) \). Thus our focus on stationary policies does not affect the practical implications of the analysis.

\(^9\)Formally, a restoration policy is characterized by a measurable closed subset \( R \) of \( \mathbb{R} \).
The manager’s objective is to choose a combined abatement–restoration policy that maximizes this expectation simultaneously for all \( x \).

3 Optimal abatement and restoration policies

In this section we characterize optimal restoration and abatement policies, and show how they interact. We then consider how the optimal policies vary with the magnitude and variability of flows, the costs of restoration and abatement, and the discount rate. Finally, we briefly consider three extensions of the basic model.

3.1 Characteristics of the optimal policies

We use stochastic dynamic programming to characterize the optimal restoration and abatement policies. Let \( J \) be the optimal value function:

\[
J(x) = \sup_{a,R} E_x^{a,R} \left[ \int_0^\infty e^{-\alpha t} (u(x_t) - c(a(x_t))) dt - \sum_{i=1}^\infty e^{-\alpha T_i} C_i \right],
\]

where the supremum is taken over pairs of abatement and restoration policies. \( J(x) \) represents the maximal present value of the future stream of net benefits (utility minus cost) under the optimal policy, starting from state \( x \).

Theorem 1 describes the optimal abatement and restoration policies, and the resulting path of resource quality. It identifies two key quality levels: \( \underline{x} \), the restoration trigger; and \( \overline{x} \), an inflection point in the value function that coincides with maximum abatement.

**Theorem 1** Let Assumptions 1, 2, and 3 hold. Then there exist states \( \underline{x} \) and \( \overline{x} \), with \( \underline{x} < \overline{x} \), such that the following results hold:

(Qualities of the value function) (i) \( J < 0 \) and \( J(x) \) is finite for every \( x \). (ii) \( J(x) = J(0) - C \) for all \( x \leq \underline{x} \). (iii) \( J \) is continuously differentiable for every \( x \), and is twice continuously differentiable on \((\underline{x}, \infty)\). (iv) For all \( x > \underline{x} \), \( J \) satisfies

\[
\sup_{a \in [0, \overline{a}]} \left( \frac{\sigma^2}{2} J''(x) + (a - \mu)J'(x) - \alpha J(x) + u(x) - c(a) \right) = 0. \tag{4}
\]

(Shape of the value function) (v) \( J'(x) > 0 \) for \( x \in (\underline{x}, \infty) \). Moreover, (vi) \( J''(x) > 0 \) for \( x \in (\underline{x}, \overline{x}) \); (vii) \( J''(\overline{x}) = 0 \); and (viii) \( J''(x) < 0 \) for \( x \in (\overline{x}, \infty) \).
(Optimal policy) (ix) There is a function $a^*: (\underline{x}, \infty) \mapsto [0, \overline{a}]$ such that for every $x \in (\underline{x}, \infty)$, $a^*(x)$ uniquely attains the supremum in equation (4). (x) $a^*$ is increasing on $\{x \in (\underline{x}, x^\dagger) | a(x) \neq \overline{a}\}$ and decreasing on $\{x \in (x^\dagger, \infty) | a(x) \neq \overline{a}\}$. (xi) Letting $R^* = (-\infty, \underline{x}]$, the pair $(a^*, R^*)$ is an optimal policy.

This theorem (along with subsequent ones) is proven in Appendix A. Here we discuss the optimal abatement and restoration policies established by the theorem. Figure 1 illustrates optimal policies for a particular set of parameters. (The functional forms and parameter values used for all figures are provided in Appendix B.) Quality $x$ is plotted on the horizontal axis. The optimal abatement rate, $a^*(x)$, appears above the axis, with the corresponding values of the value function $J$ below.

Under the optimal policy, the manager restores the resource whenever quality falls to $\underline{x}$. This closely resembles the familiar solution to the classic inventory problem: a profit-maximizing firm will follow an $(S, s)$ rule in managing its inventory, drawing its stock of goods down until some level $s$ is reached and then replenishing the inventory up to the level $S$ (Arrow, Harris, and Marschak 1951; Scarf 1960). The “inventory” in the restoration case is quality, and a restoration corresponds to a replenishment of inventory.

As Figure 1 shows, the optimal abatement rate varies with the state $x$: first rising as quality worsens, and then falling smoothly to zero at the restoration point $\underline{x}$. This policy can be understood heuristically as equating marginal benefits and marginal costs at each level of quality $x$. From Theorem 1, the abatement rate must attain the supremum of a function $f_x(a) = aJ'(x) - c(a)$ (the components of equation (4) that are a function of $a$). The first term, $aJ'(x)$, represents the rate at which the value function increases. This corresponds roughly to the expected benefit from abating at rate $a$, taking into account present and future utility. The second term, $c(a)$, represents the cost of abatement $a$. Hence the optimal policy at each state sets the abatement rate to maximize the resulting “expected net benefit.”

Because utility is concave, the marginal benefit from abatement at first increases as $x$ diminishes. The optimal rate of abatement rises accordingly. Abatement reaches its peak at a state $x^1 > \underline{x}$, where the marginal benefit of abatement is greatest. Beyond this point, the optimal abatement

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10 Heuristically, for a given marginal change in the state $dx$, the resulting change in the value function would be $J'(x)dx$. Abatement $a$, carried out over an infinitesimal time period of duration $dt$, yields a marginal improvement in the state due to abatement $dx = adt$. We can think of $(adt)J'(x)$ as the resulting change in the value function (over an infinitesimal period of time). Dividing through by $dt$ yields the rate of change in the value function, $aJ'(x)$. (Note that this heuristic explanation, like subsequent ones, puts intuition ahead of rigor, and thus is less technically precise than the formal results it seeks to explain.)

11 The key state $x^1$ marks an inflection point in the value function: $J$ is concave above above $x^1$ and convex
path reverses course, with abatement decreasing as quality continues to worsen. As the trigger point $x$ nears, the marginal benefit of abatement diminishes, since the quality of the resource will soon be restored.\textsuperscript{12}

Evidently, if the abatement rate rises high enough, it will equal or exceed the average flow rate $\mu$. Suppose this condition obtains, and let $x^*$ denote the highest quality level at which $a(x^*) = \mu$. (See Figure 1.) We call $x^*$ an expectation equilibrium: since abatement at $x^*$ equals the expected flow, the quality level will remain there in expectation.\textsuperscript{13} Moreover, $x^*$ is locally stable. Starting from $x^*$, a “high” flow of deterioration (at a rate greater than $\mu$) will depress environmental quality below $x^*$. In response, abatement will increase, and the quality level will return to the target $x^*$ in expectation. A “low” flow of damages will raise quality above the target level, leading to a slackening of abatement efforts and a tendency back toward $x^*$.\textsuperscript{14} Moreover, the expectation equilibrium satisfies $J'(x^*) = c'(\mu)$: it occurs at the point where the marginal cost of fully abating expected new pollution (achieving a zero net flow in expectation) just equals the marginal net benefits from doing so. This result is analogous to the steady-state equilibrium derived in deterministic optimal control models. Nonetheless, as we show in the next section, restoration remains a remote possibility and hence still affects the optimal abatement path.

\subsection*{3.2 The interdependence of abatement and restoration}

While the optimal restoration and abatement policies share features with familiar models, neither strategy takes the form it would in the absence of the other. First, consider how the possibility of abatement affects optimal restoration. The availability of abatement makes the resource more valuable than it would otherwise be, because its deterioration can be slowed. Hence the value

\footnote{The convexity of the value function below $x^\dagger$ – despite the concavity of the underlying utility function – is a consequence of the optimal restoration policy. $J$ is constant below $x$, since the restoration always returns the state to $x = 0$ at a fixed cost. Because $J$ is differentiable, its slope at $x$ is zero. Above $x$, $J$ is increasing. In some region just above $x$, therefore, $J(x)$ must be convex. The upper bound of this region is the inflection point $x^\dagger$.}

\footnote{Note that the abatement rate falls to zero at the restoration trigger point $x$. The marginal benefits of further abatement at that point are zero, because the state will be restored to $x = 0$ immediately. The smooth-pasting condition (Krylov, 1980) requires that as the state approaches the trigger, the marginal benefits from abatement decline smoothly to zero. Marginal cost must follow suit, implying that abatement must go to zero as well.}

\footnote{This result would change slightly under the more general cost functions considered in section 3.4.2. Given a variable component of restoration cost $\gamma(x)$, with $\gamma'(x) < 0$, the smooth-pasting result would still hold; but abatement would decline smoothly to $a > 0$ satisfying $c'(a) = -\gamma'(x)$.}

\footnote{We use the term “equilibrium” in the sense of “system stability,” rather than in a strategic or game-theoretic sense. The “expectation” refers to the frequency distribution of states, rather than the beliefs of economic agents.}

\footnote{Because $a$ declines to zero as the trigger level approaches, there must be a second state $x^{**} < x^\dagger < x^*$ at which optimal abatement again equals the average flow rate. This too is an equilibrium, but it is unstable: any deviation in flow from $\mu$ will tend to push quality either downward toward $x$ or upward toward $x^*$. If flows are large enough to push quality below $x^{**}$, the state will (again in expectation) decline to the restoration trigger $x$.}
function when abatement is possible must lie above where it would if only restoration were possible. Moreover, the magnitude of the shift is not the same everywhere. An immediate implication is that the availability of abatement affects the optimal restoration trigger.

Intuition might suggest that restoration and abatement should be substitutes for one another: that is, one might expect the availability of abatement to lower the optimal restoration trigger. In fact, however, the trigger level can go either up or down. When quality is high, restoration is distant, and abatement allows the manager to maintain quality at a higher level than would be possible in its absence. Hence introducing abatement may raise the value function more at high levels of quality than at low levels. If so, the restoration trigger must rise. Figure 2 presents such a case. Different circumstances, however, can produce the opposite result. Simulations demonstrate that the availability of abatement lowers the restoration trigger when mean flow is sufficiently low.

The effect of restoration on abatement, on the other hand, is unambiguous: less abatement is optimal when restoration is feasible. This result is stated in the following theorem.

**Theorem 2** Let Assumptions 1, 2, and 3 hold. Let \( J_{\text{abate}} \) and \( a_{\text{abate}} \) denote the optimal value function and abatement policy in the absence of restoration. Then (i) \( J' < J'_{\text{abate}} \); and (ii) for each state \( x \in (x, \infty) \), where \( x \) is the trigger point under restoration, either \( a(x) < a_{\text{abate}}(x) \) or \( a(x) = a_{\text{abate}}(x) = \overline{a} \).

A key consequence of restoration is to reduce the rate of change of the value function, relative to the case where only abatement is possible. The feasibility of restoration must raise \( J(x) \) everywhere, since its absence represents a constraint on the resource manager. But the value function increases more at low levels of quality, where restoration is imminent, than at high levels of quality, where restoration is more distant. Since \( J'(x) \) is smaller when restoration is possible, the marginal benefits from abatement are also lower, so that less abatement is optimal. A corollary is that even if an expectation equilibrium exists when restoration is feasible, it must occur at a lower level of quality. That is, \( x^* < x^*_{\text{abate}} \). Thus the availability of restoration alters the optimal abatement policy, even when the probability of its occurrence is very low.

Figure 3 summarizes this discussion, portraying optimal abatement policies with and without the possibility of restoration. In the top panel, an expectation equilibrium exists when restoration is available, but it occurs at a lower level quality than if restoration were not an option. The bottom panel of Figure 3 illustrates a case in which restoration has a more drastic consequence for the abatement path. No expectation equilibrium exists: at all values of \( x \) above \( \overline{x}, \) abatement
merely slows – but never halts – the net flow of damages. Rather than maintaining quality at a
certain level, the optimal policy lets damages accumulate until the trigger level is reached, and then
restores the resource. Which of the two cases portrayed in Figure 3 prevails depends on how fast
the resource deteriorates, as we discuss in the next section.\footnote{A third possibility exists: the optimal abatement rate may equal average flow at exactly one point, so that the
abatement function is tangent to the horizontal line at $\mu$. The resulting “expectation equilibrium” $x^*$ is then stable
from the right but not the left. For $x > x^*$, abatement will be lower than $\mu$ and the state will tend to return to
$x^*$. For $x < x^*$ abatement will also be less than $\mu$, implying that in expectation the quality level will decline to the
trigger level for restoration.}

3.3 The roles of flow characteristics and control costs

Next, we consider how the optimal policies depend on the magnitude and variability of flow, the
costs of control, and the discount rate. We highlight how these factors affect the optimal mix of the
two methods of meliorating quality. We rely on simulations for most of our results, since important
relationships in the model are often complex, hence resistant to straight analytic demonstrations.

3.3.1 Effects on the restoration trigger

The restoration trigger for each abatement path is the point at which abatement falls to zero. Figure
4 plots the restoration trigger against the flow rate $\mu$ and the variance rate $\sigma^2$, for a particular
range of parameter values. The top two panels represent two-dimensional projections, showing how
the trigger varies with one parameter or the other. The lower panel shows the three-dimensional
surface over the same range. The restoration trigger is measured on the vertical axis; notice that
$\sigma^2$ increases “out of” the page along the $x$-axis, while $\mu$ increases from left to right along the $y$-axis.

When flows are sufficiently high, the restoration threshold is inversely related to both the flow
rate and its variability. That is, higher flows and greater variability both drive the restoration trigger
down. This result can be understood heuristically by analogy with the theory of real options.\footnote{See Dixit and Pindyck (1994) for a thorough discussion of option value in the context of dynamic stochastic
models of investment under uncertainty.} Recall from Theorem 1 that the value function is convex just above the restoration trigger, and flat
below it. Since the rate of deterioration is stochastic, and restoration is an irreversible investment,
there is an option value to waiting before restoring. Just above the trigger, a favorable shock raises
quality and expected utility. The “downside risk,” however, is limited, since the cost of restoration
is unaffected by quality. The resulting option value represents a reward to waiting, and so leads to
a lower restoration trigger. Moreover, the reward is greater when flows are more variable.
These effects are reversed at lower flow rates, however (not shown in Figure 4). When flows are low, an expectation equilibrium exists, interrupting the decline of quality towards the restoration trigger; this may alter the relationship between flows and restoration. We conjecture the following: when flows are sufficiently high that an expectation equilibrium does not exist, the restoration trigger decreases monotonically with the mean flow rate and with the variability of flows.\textsuperscript{17}

The relationship between abatement cost and the restoration trigger is also ambiguous. The effect of restoration cost, of course, is clear-cut. The next theorem states that as the cost of restoration increases, the trigger level $\bar{x}$ decreases. As intuition would suggest, a higher cost increases the incentive to delay restoration.

**Theorem 3** Let $\bar{x}(C)$ denote the trigger level for a given restoration cost $C > 0$. Then $\bar{x}(C)$ is decreasing in $C$.

The restoration trigger also appears to decline monotonically with the discount rate. That is, the more weight is put on current rather than future welfare, the further the resource is allowed to deteriorate before restoration occurs.

### 3.3.2 Effects on abatement

Figure 5 illustrates optimal abatement policies for three values of the mean flow rate $\mu$. Panel (a) depicts the absolute abatement rates, while panel (b) depicts the abatement rates measured as fractions of the mean flow, so that abatement is normalized by flows. When the mean flow rate is high, the cost of offsetting it with abatement is high as well. At the same time, restoration will be more frequent, on average, so that damages will persist for a shorter period of time before the resource is restored. Hence at higher flow rates, restoration becomes more attractive relative to abatement, and less abatement is done.

Greater variability has a similar effect on the optimal abatement path. Figure 6 depicts the effects of the variance rate $\sigma^2$ on the optimal abatement path. Note that abatement reaches a higher maximum when the variability is lower. This is a consequence of Jensen’s Inequality. Because abatement costs are convex, abatement becomes more expensive in expectation as the variability rises, and thus less abatement is done.

\textsuperscript{17}Our conjecture is a statement of sufficiency, not necessity. Simulations show that the trigger falls with variability even in some cases with expectation equilibria. Indeed, for the lowest flow depicted in Figure 4 ($\mu = 1.0$), an expectation equilibrium exists for almost all the entire range of $\sigma^2$. 

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Figure 7 shows how the importance of abatement relative to restoration varies with the flow rate. The horizontal axis measures the flow rate. The vertical axis measures the time-averaged rate of abatement as a fraction of the flow of deterioration, or (equivalently) the fraction of total melioration optimally achieved by abatement rather than restoration. Thus for a given flow rate, the height of the curve represents the fraction of melioration due to abatement. The remainder, from the curve to the top of the graph, is due to restoration. For example, at a flow rate of 3, abatement accounts for approximately 20% of melioration, and restoration 80%.

Figure 7 also illustrates how the existence of an expectation equilibrium depends on the flow rate. The dashed line on the figure marks the critical value of the flow rate that determines whether or not an expectation equilibrium exists. (In Figure 7, this critical value is just above 1.2.) For flow rates below this critical value, abatement offsets expected deterioration completely over some range, so that an expectation equilibrium is reached. In this case restoration occurs with very small probability. For low flow rates, then, we may say that abatement is the “principal melioration strategy.” When flows increase beyond this cutoff, the expectation equilibrium vanishes, and restoration becomes the principal strategy.

Similarly, restoration predominates as flows become more variable. Plotting the fraction of melioration by abatement against the variance rate $\sigma^2$ would reveal a similar pattern as in Figure 7. Since greater variability of flows depresses the optimal abatement rate, the share of melioration achieved by abatement falls as the variability rises.

An interesting feature of the SFQ model is that the optimal balance between restoration and abatement is highly sensitive to small changes in key parameters. Consider the effects of changes in the flow rate around the critical value of 1.2 in Figure 7. As the mean flow rate increases from 1.1 to 1.3, the fraction of melioration by abatement drops dramatically, from 0.9 to 0.6. Between $\mu = 1$ and $\mu = 2$, the abatement fraction falls from 0.97 to 0.33. The optimal mix of the two strategies is similarly sensitive to changes in variability, marginal abatement cost, and the discount rate around the critical values that determine whether an expectation equilibrium exists.

Whether the optimal policy relies more on restoration or on abatement determines how the quality of the resource varies over time. Figure 8 plots the frequency distributions of states for three flow rates. When flows are low, an expectation equilibrium is achieved. States close to this equilibrium level are much more common than other states. Indeed, the peak of the frequency distribution for $\mu = 1$ occurs precisely at the expectation equilibrium ($x = -185$) depicted in Figure 5 for the same flow rate. At somewhat higher flow rates, no expectation equilibrium exists,
and restorations occur more frequently. As a result, high-quality states become relatively more common, flattening the frequency distribution. For the moderate flow rate depicted in the figure ($\mu = 1.5$), the distribution retains a peak, occurring just above the point at which abatement reaches its maximum. At the high flow rate, restoration becomes even more important relative to abatement, and all states between the initial quality level and the restoration point occur with roughly equal frequency.

The effects of economic variables accord with intuition. When marginal abatement cost is higher, the optimal abatement rate is lower at every state, and the fraction of melioration achieved through abatement falls. When restoration is more expensive, more abatement is done.

The effect of higher discount rates, depicted in Figure 9, is perhaps more interesting. When the quality of the state is high, a higher discount rate leads to less abatement – as might be expected. As the state deteriorates, however, this relationship inverts: abatement becomes greater at higher discount rates, and thus reaches a higher peak. Putting more weight on current relative to future utility results in more abatement, because the benefits from eventually restoring the resource are discounted more heavily. This result is intuitive, and yet leads to a surprising conclusion. In conventional models of the management of resource stocks, greater patience leads to more abatement in the near term, not less. For example, in models of stock pollution, lower discount rates lead to lower emissions. In the management of fisheries, putting more weight on the future prescribes lower catch levels in the near term in order to achieve a higher steady-state stock. In contrast, in the SFQ model, over a range of states a more patient manager will be more tolerant of environmental degradation in the short run – because such tolerance will hasten restoration, and raise environmental quality in the long run.

### 3.4 Extensions

#### 3.4.1 Endogenous destinations

We close this section by describing several extensions to our results that could be proved using similar tools. The first involves freeing the initial quality level $x_0$. Our model took $x_0 = 0$ to be the fixed destination of any restoration activity, with the cost of such a restoration being a constant $C$. A more general model would allow for variable degrees of restoration that could result in any quality level $x \in [x_t, \infty)$. If we assume that the cost of such a restoration is defined by a function $C$ that maps the destination state to the cost of the restoration and is bounded below by a positive
scalar, then there will be a state $\bar{x}$ and an optimal policy that restores the state to $\bar{x}$ every time that $x_t \leq \bar{x}$ for some critical state $x < \bar{x}$. Hence, qualitative characteristics of the value function and optimal policy, as well as dynamics of the system, are virtually identical to those presented above. The only difference is that the state $\bar{x}$ to which an optimal policy restores is determined by the policy and not restricted by assumption. In particular, the results of Theorem 1 hold if Assertion (ii) is replaced by: $J(x) = J(\bar{x}) - C(\bar{x})$ for all $x \leq \bar{x}$.

### 3.4.2 Greater costs for greater restoration

We have assumed that restoration costs are “destination-driven,” in that they depend on ultimate rather than initial quality. However, the results of the model hold for cost functions exhibiting less extreme economies of scale. For example, suppose that the cost of restoring the resource to $x = 0$ starting from quality level $x$ has a fixed component $F$, as before, but also has a variable component $\gamma(x)$. We would expect $\gamma(x)$ to be a decreasing function of $x$; i.e., the restoration cost increases with the amount of restoration done. Total cost is given by $C(x) = F + \gamma(x)$. Unless $J(0) - J(x) < C(x)$ for all $x$, restoration will be optimal for at least one state $x$. If we now let $\bar{x}$ denote the highest value of $x$ at which restoration is optimal, then the system will evolve much as in the case with only a fixed cost for restoration.

If $\gamma(x)$ is convex, the restoration policy $R$ may no longer be a convex set. Nonetheless, the evolution of the system will be similar, since restoration will be triggered each time the state hits $\bar{x}$. In this case too, the extent of scale economies will clearly depend on the relative size of the fixed cost $F$, and will determine how far the state falls before restoration is undertaken.

As an alternative formulation of cost, consider a model in which restoration can result in any quality level $x \in [x_t, \infty)$, and the cost incurred is a function $C$ of the quality difference $x - x_t$. Let restoration cost increase with the amount of restoration done, but at a decreasing rate (i.e., with economies of scale): thus $C > 0$; $C' > \varepsilon$, for some positive scalar $\varepsilon$; and $C'' < 0$. Then there is once again a distinguished state $\bar{x}$ such that restoration occurs if and only if $x_t \leq \bar{x}$. Unlike the previously discussed cases, the optimal destination state generally depends on the value of $x_t$. However, in steady-state, since the process is continuous, restoration should always occur when $x_t = \bar{x}$ and there will be a state $\bar{x}$ that serves as the destination state every time this happens. In particular, the results of Theorem 1 once again hold so long as Assertion (ii) is replaced by: for each $x \leq \bar{x}$, there is a state $\bar{x}(x) > x$ such that $J(x) = J(\bar{x}(x)) - C(\bar{x}(x) - x)$; furthermore, $J(x)$ is increasing and $\bar{x}(x)$ is decreasing on $(-\infty, \bar{x})$. 

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3.4.3 Delayed restoration

In many real-world applications of the SFQ model, restoration is unlikely to be instantaneous. For example, proposed methods to remove greenhouse gases from the atmosphere (e.g., seeding oceans with iron filings to promote plankton growth) would require long lead times.

Consider a generalization of the model formulated in Section 2, where a delay of length $D$ is incurred in restoration. During the interval $[T_i + D, T_i]$ between the completion of the $i$th restoration project and the commencement of the next, the state evolves according to $x_t = \int_{s=T_i+D}^{t} ((a(x_s) - \mu)ds + \sigma dw_s)$. The optimal value function is now defined by

$$J(x) = \sup_{a,R} E_x^a,R \left[ \int_{t=0}^{T_1} e^{-\alpha t} (u(x_t) - c(a(x_t))) dt + \sum_{i=1}^{\infty} \int_{t=T_i+D}^{T_{i+1}} e^{-\alpha t} (u(x_t) - c(a(x_t))) dt - \sum_{i=1}^{\infty} e^{-\alpha T_i} C \right]$$

(compare to equation (3)). Note that states observed during a restoration project do not enter into equation (5). Instead, the restoration cost $C$ now incorporates all costs incurred and utility realized in the course of a restoration project. While the length of the delay is deterministic in this model, the restoration cost could be random (in which case $C$ would be the expected cost).

The results in Theorems 1-3 continue to hold in such a model, with only minor alterations.\textsuperscript{18}

4 Implementation when flows are decentralized

Thus far, we have implicitly assumed that the resource manager is able to implement the optimal abatement and restoration policy directly. In many cases of real-world interest, however, no single “manager” exists. For example, stock pollutants such as carbon dioxide are emitted by factories, power plants, and motor vehicles; municipal solid waste is generated by individual households; highways become potholed from the passage of cars and trucks. To achieve optimality, the central authority (usually the government) must regulate the activities of these private parties.

In this section, we consider the natural setting in which deterioration (and thus abatement) depends on a large number of self-interested agents who are subject to regulation by a central

\textsuperscript{18}Two slight modifications are required. In Theorem 1, the boundary condition stated in the first part becomes $J(x) = e^{-\alpha D} J(0) - C$ for all $x \leq \underline{z}$. In Theorem 3, the trigger level $\bar{z}$ is nonincreasing (rather than decreasing) in $C$. (With delayed restoration, if the restoration cost $C$ is sufficiently low, the optimal policy will be continually to restore the resource – effectively avoiding the negative utility that comes from letting the resource deteriorate.)
authority; that authority also carries out any restoration. Two issues arise with respect to implementation. First, how can the optimal abatement path be achieved? Second, how should the central authority support the costs of restoration?

As a motivating example, we consider emissions of a stock pollutant by an industry of \( N \geq 1 \) firms. Suppose that in the absence of any regulation, firm \( j \)'s average emissions would be \( \mu_j \) per unit time, so that the mean flow of emissions from the industry as whole is given by \( \mu = \sum_{j=1}^{N} \mu_j \).

The actual net flow of pollution might differ from \( \mu \) due to fluctuations in the decay of accumulated pollution stocks, variation in the physical variables (e.g., temperature) affecting the formation of new pollutants, or random shocks to the firm’s output. Let \( \sigma_j^2 \) denote the variance rate of firm \( j \)'s emissions; we assume that the random fluctuations at individual firms are independent, so that \( \sigma^2 = \sum_{j=1}^{N} \sigma_j^2 \). Finally, denote firm \( j \)'s abatement by \( a_j \), and its cost of abatement by \( c_j \); the aggregate abatement cost function is \( c(a) = \min \{ \sum_{j} a_j \} \sum_{j=1}^{N} c_j(a_j) \).

### 4.1 Aligning incentives for abatement

Not surprisingly, a tax on pollution (more generally, on deterioration) can achieve the optimal abatement path. Recall from equation (4) that along the optimal abatement path, \( c'(a(x)) = J'(x) \) at every state \( x > x_0 \). Consider a tax set equal to the derivative of the value function, \( J'(x) \) (representing the present discounted value of the net benefits of abatement). Faced with such a tax, each polluter will abate up to the point where the marginal cost of abatement equals the tax. Hence \( c'_i(a_i) = \tau(x) = J'(x) \), ensuring that the optimality condition \( c'(a) = J'(x) \) is met.\(^{19}\) Note that while the tax varies over time to equal marginal damages, it is linear at each point in time (more precisely, at every state \( x \)). Thus each polluter always faces a linear tax on emissions (e.g., a certain amount per ton of pollution), and all polluters face a common tax rate; but this tax rate changes over time with the quality of the resource.

That such a tax achieves the first-best outcome is not surprising. However, a few novel twists follow directly from the analysis thus far. First, the optimal tax rate will follow the same qualitative path as abatement: rising at first as quality deteriorates, then reversing course and ultimately going to zero at the restoration point. Note that if restoration were not available, an unabated unit of pollution would add to the stock at all subsequent states, reducing future utility forever.

\(^{19}\)Note that in order to ensure that incentives are properly aligned, the regulator must both levy a tax on net emissions and pay out a subsidy on net abatement. That is, the flow of revenue to the regulator, \( \tau(x_1)(\mu - a_t) \), will be positive or negative depending on \( \mu \geq a_t \). In many cases — e.g., air pollution from factories — it is natural to assume that \( \bar{a} \leq \mu \) — i.e., that abatement cannot exceed unregulated emissions. If so, the tax will always be positive.
Restoration, however, wipes the slate clean: deterioration accumulated prior to restoration has no effect on the value function afterward. As restoration nears, additional pollution will affect future utility over a shorter duration (at least in expectation), and thus impose smaller marginal damage.

Second, the optimal tax rate has a surprising relationship to the mean flow rate, $\mu$. Recall from Figure 5 that abatement may reach a much higher peak when flows are lower. Because the optimal tax rate is a monotonic function of optimal abatement, it too may reach a greater maximum when flows are low. Indeed, in our simulations a plot of the tax rate against the quality would be identical to panel (a) of Figure 5, only scaled up by a constant.\(^{20}\)

Note that in the decentralized setting, $\mu$ corresponds to the amount of pollution that would result in the absence of regulation. Since the costs of abatement are independent of the mean flow (they depend only on abatement), a higher mean flow rate corresponds to more pollution at every price. Hence the SFQ model shows that the optimal tax on pollution may be lower when there is more pollution (or potential pollution) around. Again, this result contrasts sharply with the familiar static or steady-state models, where the efficient tax increases with the unregulated level of pollution as long as abatement cost depends only on the amount abated (as we assume here). Indeed, if restoration were not available, then the optimal tax rate would be strictly increasing in the mean flow rate. In that case, the greater is the mean flow, the sooner would the state arrive at poor states, where pollution imposes greater marginal damages.

With restoration feasible, the flow rate can have a different effect on marginal damages. At high levels of quality, restoration is remote, and the dynamic remains much the same as the no-restoration case just described: A higher flow increases marginal damages from current pollution, because it speeds the arrival of states in which marginal utility is high. The situation reverses at low quality levels. Now the predominant effect is that higher flow hastens restoration, which terminates the impact of current pollution; hence marginal damage is lower. Optimal abatement, and the optimal tax rate, fall accordingly.

The fact that a rise in unregulated pollution can lower the optimal tax, at least over some range of quality, has interesting implications for how the optimal tax varies with industry output. In particular, increasing the scale of an industry may lead to a lower tax rate (whereas it would always raise the tax in the setting without restoration).

Finally, note that the presence of scale economies raises interesting possibilities for the optimal

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\(^{20}\)The scaling factor is the slope of the marginal cost function, equal to 80 in our simulations (see parameter values in Appendix B).
management of multiple resources when restoration is possible. Consider the optimal siting of polluting factories along two rivers. Because marginal damages from pollution are assumed to be increasing, the optimal policy in the absence of restoration would maintain the same level of pollution on both rivers (all else equal). When restoration is possible, however, it might be optimal instead to concentrate pollution flows in one river – leading to high pollution flows and relatively frequent restorations on that river, while keeping flows low (and quality high) on the other.

4.2 Raising funds for restoration

Next, consider the question of how the central authority should raise the funds necessary for restoration efforts. Given the stochastic nature of flows, the revenue raised by the state-dependent tax just described will differ somewhat from the funds necessary for restoration, by an amount proportional to the variance rate $\sigma^2$. To see this, recall that the cost of restoration, $C$, must equal the gain in the value function from restoration. Thus $C = J(0) - J(x) = -\int_{t=T_{i-1}}^{T_i} dJ(x_t)$, where $T_i$ (as before) is the time of the $i$th restoration. Applying Ito’s Lemma, the integral becomes:

$$C = -\left(\int_{t=T_{i-1}}^{T_i} J'(x_t) \, dx_t\right) - \left(\frac{\sigma^2}{2} \int_{t=T_{i-1}}^{T_i} J''(x_t) \, dt\right).$$

The first term on the right-hand side of equation (6) is precisely the revenue collected from a tax rate $\tau(x) = J'(x)$, levied on the observed deterioration in resource quality, i.e., on $-dx_t$. (Note that this revenue is positive.) We term $\tau(x)$ the flow tax. The sign of the second term depends on the sign of $J''$ over the realized path of the state $\{x_t\}$. Since the value function has both convex and concave regions, this second term may be positive or negative: thus the revenue from the flow tax may be higher or lower than the cost of restoration.

This additional term suggests the natural way to bring revenues in line with restoration cost: the center could collect a tax $\rho(x) = -\frac{J''(x)}{2}$, levied on the variance generated by each firm per unit time – i.e., on $\sigma^2 dt$. Note that the total revenue from such a variance tax, over all firms, would equal $-\frac{\sigma^2}{2} J''(x) \, dt$ over an interval $dt$. This tax has an intuitive interpretation. Consider the range of states where the value function is concave, so that $J'' < 0$. By Jensen’s Inequality, greater variance in that range lowers expected net benefits from the resource. Thus the variance tax $\rho$ is

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$21$ Recall that individual firm’s emissions are assumed to be independent. Note that if stochastic shocks to firms were correlated, the optimal variance tax would incorporate the covariance of a firm’s emissions with those of the industry as a whole.
positive: it represents a penalty on the polluting firms for introducing variance into the resource, with the size of the penalty a function of the reduction in expected net benefit. Over the range where $J$ is convex, variance increases expected net benefits, and $\rho$ represents a subsidy.

Theorem 4 summarizes the role of the flow and variance taxes and their relationship to the cost of restoration.

**Theorem 4** Suppose the center collects a flow tax $\tau^*(x) = J'(x)$ on the observed deterioration in resource quality, i.e., on $-dx_t$, and a variance tax $\rho^*(x) = -\frac{1}{2}J''(x)$ on variance per unit time, i.e., on $\sigma^2_jdt$. Define $Y_i^*$ as the random variable equal to the resulting total tax revenue collected by the center between the $(i-1)^{th}$ and $i^{th}$ restorations. Then $Y_i^* = C$ with probability one, for all $i$.

Note that the variance tax — and hence the difference between the restoration cost and the revenues from the flow tax alone — shrinks as the variance rate of flows gets smaller. Thus when variability is relatively small, the center could charge only the flow tax, and still raise approximately the funds needed for restoration. Simulation results indicate that for the parameter values considered throughout Section 3 of this paper (i.e., $\sigma^2 = 9$ and $\mu \in [1,3]$) the discrepancy is on the order of 1% of the restoration cost or less. For $\sigma^2 = 1$, the difference is on the order of 0.1%.

Indeed, in the limiting deterministic case, a single instrument — the flow tax — achieves both of the center’s aims: it provides efficient abatement incentives and raises precisely the amount of revenue needed to pay for restoration.

Thus far in the analysis, we have assumed that the center collects the flow tax on the actual change in the state, even though the state evolves stochastically. Hence the industry rather than the central authority bears the risk from random fluctuations in quality. If individual firms are risk averse, the center may prefer to charge them only for the expected flow, given by $\mu - a_t$. In this case, the total tax revenue from the flow and variance taxes will no longer be equal to the restoration cost in general. For example, if a series of negative shocks led to unusually rapid deterioration, the center might collect less than $C$ in tax revenue. Nonetheless, the next theorem establishes that the tax revenue would still equal the restoration cost in expectation.

**Theorem 5** Let $\bar{Y}_i$ be the total tax revenue collected by levying the flow tax $\tau^*(x)$ on expected emissions given abatement efforts, equal to $(\mu - a_t)dt$, along with the variance tax $\rho^*(x)$ on $\sigma^2_jdt$. Then $E[\bar{Y}_i] = C$.

Finally, consider the effects of the variance tax on the incentives of the polluting firms. First, suppose that the variance of the flow is exogenous for every firm: that is, $\sigma^2_j$ is taken as given.
This is implicitly what we have assumed thus far in the analysis. In the pollution case, it might correspond to a model in which environmental processes affect how emissions translate into air quality.\textsuperscript{22} In this case, the flow tax $\tau(x)$ aligns abatement incentives properly, while the variance tax $\rho(x)$ is purely revenue-raising (and thus nondistortionary).\textsuperscript{23}

The situation becomes more interesting if variance is endogenous. For example, the variance of pollution flows might depend on shocks to demand for the firm’s output, or on the costs of inputs such as materials or labor, or on the characteristics of fuel inputs. In this case, $\rho(x)$ gives firms efficient incentives to adjust the variance of their flows. Suppose first that firms can make a one-time adjustment to variance, e.g., by modernizing their factory equipment or improving the consistency of their fuels. Then the variance tax would provide the right incentives to choose variance, balancing the expected benefits of higher or lower variance against the costs of adjustment. Alternatively, consider a model in which firms can dial the variance of emissions up or down, e.g., by shifting production schedules. In that case, efficient policy would call for firms to lower their variance when quality is high, but raise it as restoration neared.

Note that the presence of restoration makes a critical difference in the form of the tax policy. If restoration were not possible, the value function would be concave everywhere, and the variance tax $\rho$ would always be negative. In contrast, the availability of restoration makes variance desirable, at least over some range. The intuition in this case is as follows: When flows are more variable, we are more likely to have recently restored the resource at any point in time, since the restoration threshold is more likely to have been crossed. Hence the greater the variance, the more likely is the resource to be at a high level of quality.

We can establish a useful result in the scenario in which only restoration is possible. In this case, variance always raises the value function, as long as the resource manager is sufficiently patient. That is, the value function is increasing in $\sigma^2$ for small enough values of the discount rate $\alpha$. If the discount rate is sufficiently low, this exchange of more numerous cleanups for higher quality makes variability a welcome companion: it raises the present value of expected utility. This result is stated in the following theorem.

\textsuperscript{22} Although not a stock pollutant, smog provides an example where variation is exogenous. Smog results when nitrous oxide (NO\textsubscript{x}) emissions, from motor vehicles and factories, are mixed with volatile organic compounds (VOCs) produced largely by biological sources such as forests, in the presence of sunlight. Given a certain amount of NO\textsubscript{x} emissions, air quality depends on variables such as temperature, sunlight, and the biogenic production of VOCs.

\textsuperscript{23} In fact, the variance tax will have efficiency implications if there is free entry and exit into and out of the polluting industry. Here, however, the variance tax will ensure that entry and exit are socially optimal, since it will properly dock each firm for the additional loss in expected utility caused by the variance it introduces.
Theorem 6  Let Assumptions 1 and 3 hold. Let $J_{\text{restore}}(\cdot, \sigma^2, \alpha)$ be the optimal value function given variance rate $\sigma^2$ and discount rate $\alpha$, when only restoration is feasible. Then, for any $x$, there exists a scalar $\alpha^* > 0$ such that for any $\alpha \in (0, \alpha^*)$, $J_{\text{restore}}(x, \sigma^2, \alpha)$ is increasing in $\sigma^2$.

In the full SFQ case, when both restoration and abatement are available, variance is neither always desirable nor always detrimental. The proposed variance tax, changing with the shape of the value function, provides the correct incentives for firms to adjust their variances optimally.

5 Applications

Here we explore the implications of our theoretical model for the management of resource stocks in the real world. First, we briefly sketch the model’s application to a number of environmental problems, ranging from accumulating waste to zebra mussels. We then discuss how the model would apply to public infrastructure – in particular, highway construction and maintenance.

5.1 Environmental quality

The accumulation of wastes at disposal sites or generating facilities is a canonical SFQ problem. Consider the optimal management of municipal solid waste, for example. The environmental quality of a landfill site and the surrounding area diminishes as solid waste accumulates. The flow of waste may be slowed through recycling, composting, or waste reduction. Eventually, the landfill is capped, the site is restored – perhaps becoming a park or recreation area – and quality returns to its initial high level.\footnote{With solid waste management, successive waves of accumulation and restoration take place on a series of dump sites, as opposed to the cyclical cleansing and soiling of a single resource. Our model could be extended to accommodate the multiple-site case by having the exposure costs and restoration costs rise as we move to successively more expensive landfills. Essentially, this would append results from the theory of nonrenewable resources to our models (Dasgupta and Heal, 1979; Hotelling, 1931). Abatement today would be influenced by the shadow price of future restorations.} In a typical scenario, waste diversion remains roughly constant over time, or changes only with changing preferences (i.e., a desire to increase levels of recycling) or prices (e.g., land becomes more expensive, or recycled materials become more valuable). Optimal waste management, on the other hand, would vary the rate of abatement over time. When a landfill is first opened, diversion should be relatively high. That is because waste dumped early will be around for nearly the landfill’s entire life. Thus, the discounted expected damages it imposes will be high relative to the damages from garbage arriving later. As the landfill nears capacity, waste diversion should drop, since the waste will impose damages only for the brief time until restoration.
Similar issues, on a different time scale, are involved in the management of hazardous wastes. Consider the chemistry department at Harvard University.\textsuperscript{25} The department’s laboratories accumulate a variety of toxic and reactive substances. Storing such substances on campus heightens health and fire hazards.\textsuperscript{26} Removing the wastes for permanent disposal — “restoration,” in this context — involves economies of scale, reflecting the fixed costs of labor and transportation. Chemical wastes are hauled away in “lab packs”: containers are collected from labs and packed in larger drums with wastes of similar types. A 55-gallon drum of corrosive flammable liquids costs $320 to ship; a 30-gallon drum costs $215, and a single 5-gallon container $95.

At least in principle, several methods exist to control the flow of lab waste generated: experiments could be curtailed or altered to conserve chemicals; technicians could exert greater effort to prevent spills; laboratories could manage their inventories more efficiently; or some fraction of the waste stream could be purified and reused rather than thrown away, albeit at significant cost. Although some abatement would likely be optimal, little concerted effort is actually made to curb flows. A partial explanation is that until recently, individual laboratories were not charged for disposal, and thus had little incentive to reduce their chemical use. Individual labs recently began to pay a volume-based charge for both solvents and lab packs. Limited experience indicates that the use of chemical wastes is fairly inelastic, suggesting high costs of substantial abatement.

The sedimentation of reservoirs presents a very different application.\textsuperscript{27} The “stock” in this context is the capacity of the reservoir, which is diminished as sediment flows into the reservoir and accumulates. Dams are commonly designed to have finite lives: the reservoir behind a dam fills up with sediment, until the dam is retired and a replacement dam is constructed. Retirement and replacement constitute an extreme form of restoration, and one whose costs are essentially destination-driven. Alternatively, the stock of sediment can be removed directly by siphoning or dredging – activities likely to exhibit economies of scale. Meanwhile, a range of strategies exists to abate the sediment flow. The flow into the reservoir can be reduced by soil conservation, reforestation, and other measures in the catchment area; or sediment can be routed away from the reservoir. The precise nature of the optimal policy will depend on site-specific costs of stock and

\textsuperscript{25}We thank Henry Littleboy, Health and Safety Officer (for Harvard’s Faculty of Arts and Sciences Office of Environmental Health Services), who oversees hazardous waste management in the Chemistry Department, and Dr. Alan Long, Director of Laboratories, for their generosity in answering questions and providing information about hazardous waste management in the Harvard chemistry department.

\textsuperscript{26}Of course, chemical waste storage and disposal are heavily regulated by the Environmental Protection Agency. For example, existing regulations prohibit the storage of waste longer than ninety days. At Harvard, the constraint does not bind: limited storage space makes more frequent collection necessary.

\textsuperscript{27}The information about dam sedimentation is taken from Palmieri, Shah, and Dinar (2001).
flow controls. Nonetheless, the common practice of letting sediment accumulate unchecked before retiring the dam – equivalent to a restoration-only policy – is almost surely suboptimal. At the same time, “sustainable management” that seeks to maintain an equilibrium by relying exclusively on controlling sediment flow, without periodic restoration (Palmieri, Shah, and Dinar 2001), is equally unlikely to be optimal.

The SFQ model also applies to the control of animal pests, such as zebra mussels (*Dreissena polymorpha*). These small freshwater mollusks were introduced to the U. S. accidentally, carried in bilge water of cargo ships. They clog water intake and distribution systems by adhering in large clusters to pumphouses, plumbing systems, and other pieces of equipment. The control of zebra mussels by power plants, water works, and other large users of water in the Great Lakes area of the United States has been estimated to have cost as much as $1 billion in the 1990s alone.\(^{28}\)

The feasibility of preventing mussel settlement varies by location, and control strategies vary accordingly. In the pumphouses of power plants, mussels grow on walls, debris screens, valves, and pumps, obstructing the flow of water. Mechanical measures to remove them – physical scraping or “hydrolasing” with high-powered water hoses – involve high fixed costs: sending down a team of divers or even dewatering the pumphouse (thus shutting down the plant). An \((S,s)\) policy is followed. Mussels are allowed to settle and grow, and periodically are removed. Removal is done every year or two in western Lake Erie, their densest habitat. Inside the plumbing systems of power plants and water works, mussels are inaccessible to mechanical removal, but chemical removal is feasible. In such locations, both flow and stock controls are employed. Continuous low-level chlorination of circulating water is an abatement policy. It kills juvenile mussels, and inhibits their settlement. Periodic (annual or semi-annual) injections of high concentrations of chlorine represent a restoration strategy used to kill off adult mussels that have settled.\(^{29}\)

### 5.2 Public infrastructure

More generally, the basic SFQ model applies to a range of public infrastructure projects. Consider, for example, highway construction and maintenance. Highway expenditures represent a substantial

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\(^{28}\)Personal communication, Charles O’Neill, Project Director, National Zebra Mussel Information Clearinghouse, New York Sea Grant.

\(^{29}\)Continuous chlorination is typically effective enough that additional periodic treatments are unnecessary. This may be seen as an instance where flow control measures maintain a high-quality expectation equilibrium, and restorations are extremely rare. Nonetheless, if zebra mussel settlement were to occur (due, perhaps, to a breakdown in the chlorination regime), a one-time injection of chlorine at higher concentrations would be employed as a restoration measure.
chunk of public spending. Total government spending on highways in the United States averaged $122.5 billion annually over the period 1998-2002; half of that amount was for capital outlays (new investment), while a quarter was spent on maintenance (USDOC, 2004). Looking across countries, there is considerable evidence that maintenance is underfunded relative to new investment, particularly in developing countries (see the discussion and citations in Rioja (2003)). Recent studies have considered the optimal allocation of public funds for investment and maintenance in infrastructure in the context of optimal growth models (Rioja 2003; Kalaitzidakis and Kalyvitis 2004). These analyses emphasize the importance of the marginal productivity of public capital, and the degree to which public expenditure complements private investment or crowds it out.

An SFQ approach introduces restoration with economies of scale, representing replacement and/or rehabilitation of existing infrastructure.\footnote{In models of infrastructure, deterioration is typically assumed to be proportional to the existing capital stock, whereas our formulation assumes that it is independent of the stock. However, this difference can be easily accommodated. In the infrastructure setting, our model can be considered to apply to “single projects” subject to maintenance and restoration. In a model of a large number of such projects, each of identical size, aggregate deterioration would indeed be proportional to the number of projects, i.e., the total capital stock.} Our analysis points to the importance of the rate of unchecked deterioration in determining the optimal policy. The greater is economic activity, all else equal, the faster will infrastructure deteriorate, and the greater should be the expenditure on new investment (restoration) rather than maintenance. One possible implication is that the share of expenditures allocated to maintenance relative to new construction should be higher in less-developed countries (where economic activity is lower) – the reverse of the pattern observed.

Note that studies of public infrastructure spending are typically national in scale, while our analysis so far has concentrated on single projects. Suppose, however, that a large number of infrastructure projects coexist; that these projects were initiated at different points in time; and that they share a common level of unchecked deterioration. Then the aggregate state of infrastructure projects will be described by a frequency distribution of the type illustrated in Figure 8, while the aggregate share of expenditure on maintenance versus restoration at any point in time will be described by Figure 7.

Finally, the SFQ analysis provides insight into the optimal financing of infrastructure projects. Macroeconomic studies such as Rioja (2003) and Kalaitzidakis and Kalyvitis (2004) assume (with most such growth models) that projects will be financed by public funds, e.g., from a tax on output. However, our results show that a tax on deterioration could not only align private incentives with social welfare, but could also raise the revenue necessary to cover the costs of restoration projects.
For example, in the case of highways, the central authority might levy a tonnage tax on trucks. Indeed, current gasoline taxes in the U. S., which help pay for highway construction, are (at least qualitatively) appropriate instruments. Moreover, implementation of a deterioration tax would be relatively easy, given the existing structure of gasoline taxes, tonnage charges, and turnpike tolls.

6 Conclusion

In a wide range of settings, both stocks and flows can be controlled to promote the quality of a public good. If so, the SFQ model applies. Managing the resource entails abating the downward drift in quality and periodically restoring the stock. These strategies are interdependent. The optimal balance between them depends on rate and variability of ongoing deterioration, the costs of the two strategies, and the discount rate. If flows are low enough or abatement is sufficiently inexpensive, an “expectation equilibrium” may be reached where abatement efforts just offset the expected deterioration of the resource. In that case, abatement is the principal management tool, although the optimal abatement rate is still lowered by the potential for restoration. When deterioration is more rapid or more variable, when abatement is more expensive, or when restoration is less costly, the optimal policy relies more on restoration.

This model has broad relevance for the management of public goods in the real world. We have discussed a range of applications: the disposal of municipal solid waste and hazardous laboratory waste; the slowing of siltation in reservoirs; the control of pests such as zebra mussels; and the construction and maintenance of public infrastructure projects such as highways. Note that these problems involve private decisions (lab wastes), public decisions (highway infrastructure), and situations where the public and private sectors interact, often in a regulatory context.

The analysis generalizes readily to the management of physical and human capital. Optimal policies for replacing a machine, investing in capital equipment, or purchasing consumer durables cannot be derived independently of the optimal maintenance paths. Indeed, how far the productivity of capital should be allowed to fall before replacement depends not only on how costly is replacement but also on how rapid and variable is deterioration. Similarly, from the perspective of the firm, investment in human capital presents an SFQ problem. Workers age, tire, and burn out. In industries with rapid technological advance, workers’ skills quickly obsolesce. A firm can train its workers to maintain their productivity, but at some point it may lay off its older workers, or reassign them to tasks where the latest technical skills are less essential and replace them with
recently trained workers. In this context, training is “abatement,” and replacement – which incurs costs such as severance payments or raised experience-rated unemployment insurance – amounts to restoration.

Government will likely play a role in many SFQ problems, e.g., in controlling environmental quality, and must understand the central lesson: that stock and flow controls should be coordinated and implemented jointly when both are feasible. When restoration is an option, maintaining a resource stock at a constant level (by abating flows) will be more expensive than achieving the same present value of expected utility from quality, but allowing quality to vary over time. A policy relying solely on restoration will not only restore too frequently (since deterioration is unchecked), but will also allow quality to fall too far before each restoration (since the optimal trigger rises when abatement is available).

These errors are likely to be relevant to real-world environmental policies that utilize only one melioration strategy or the other. For example, environmental policies towards hazardous waste tend to emphasize terminal cleanup and permanent storage (restoration) rather than slowing waste generation. The Harvard example is instructive here, as for years the university focused almost exclusively on hauling wastes away and largely overlooked methods for curbing their generation. Regulation of air and water pollution, on the other hand, tends to focus on emissions rather than the resulting quality level in the environment. Where only flows matter, or restoration is unavailable, such an emphasis is optimal. But when pollution accumulates, policies should adjust if restoration is a possibility, either technologically or politically. For example, imagine that an economic technology is developed (some time in the future) to remove carbon dioxide from the atmosphere. In such a scenario, optimal abatement of carbon dioxide emissions would fall, and restoration would eventually take place if carbon dioxide levels climbed sufficiently.
References


Appendix A: Proofs of Theorems

Proof of Theorem 1

Proof. We have \( J < 0 \) because \( u < 0, c > 0, \) and \( C > 0. \) Furthermore, for each \( x, J(x) > -\infty \) because \( E_x \int_0^\infty e^{-\alpha t} u(x_t) dt \) is finite. We have established Assertion (i).

Because restoration sets the state to 0 and costs \( C, J(x) \geq J(0) - C \) for all \( x, \) and an optimal policy \( R \) can be defined to be the set of all \( x \) such that \( J(x) = J(0) - C. \) Let us establish that any optimal policy \( R \) is nonempty – that at some level of environmental quality the manager restores the resource. Assume, for contradiction, that the optimal restoration policy \( R \) is empty. Then, we would have \( J(x) = \sup_a E_x^a \int_0^\infty e^{-\alpha t} (u(x_t) - c(a(x_t))) dt. \) It is easy to see that \( J \) would be unbounded below, contradicting the fact that \( J(x) \geq J(0) - C. \)

By straightforward sample-path arguments, it is easy to show that \( J \) is continuous and nondecreasing. Hence, there exists a state \( \underline{x} \) such that \( J(x) = J(0) - C \) for all \( x \leq \underline{x} \) and \( J(x) > J(0) - C \) for all \( x > \underline{x}, \) establishing Assertion (ii).

It follows from Theorem 3 on page 39 of Krylov (1980) that \( J \) is twice continuously differentiable on \((\underline{x}, \infty))\) and differentiable everywhere. Furthermore, \( J \) satisfies

\[
\sup_{a \in [0, \bar{a}]} \left( \frac{\sigma^2}{2} J''(x) + (a - \mu)J'(x) - \alpha J(x) + u(x) - c(a) \right) = 0
\]

for all \( x > \underline{x}. \) Hence, Assertions (iii) and (iv) are valid. It is easily verified by sample-path arguments that \( J \) is increasing on \((\underline{x}, \infty))\) (Assertion (v)).

It follows from Assertions (ii) and (iii) that \( J'(x) = 0. \) Since \( J'(x) > 0 \) for all \( x > \underline{x}, \) we have \( J''(x) > 0 \) on some range \( x \in (\underline{x}, y) \) for some \( y > \underline{x}. \) Furthermore, since \( J \) is bounded above, \( J''(x) \) must be negative for some \( x > \underline{x}, \) and by continuity of the second derivative, there is a well-defined minimal inflection point \( x^\dagger = \min\{x > \underline{x} | J''(x) = 0\}, \) which by definition satisfies Assertions (vi) and (vii).

Now consider an optimal policy. Assertion (ii) implies that the restoration component of an optimal policy is given by \( R^* = (-\infty, \underline{x}] \). Let a function \( f_x \) be defined for \( x > \underline{x} \) by \( f_x(a) = aJ'(x) - c(a). \) Note that \( f''_x = -c'' \leq -\epsilon \) for some \( \epsilon. \) Hence, for any \( x, \) the supremum

\[
\sup_{a \in [0, \bar{a}]} f_x(a)
\]

is uniquely attained by some \( a \in [0, \bar{a}]. \) For each state \( x > \underline{x}, \) let \( a^*(x) \) be the value attaining the
supremum, and note that \((a^*, R^*)\) constitutes an optimal policy since the values \(a^*(x)\) also attain the supremum in the Hamilton–Jacobi–Bellman equation (equation 4). This validates Assertions (ix) and (xi). Moreover, for any \(x, y \in (x, x^\dagger)\) with \(x < y\), \(f'_y(a^*(x)) > f'_y(a^*(x)) = 0\), since \(J'' > 0\) on \((x, x^\dagger)\). Consequently, unless \(a^*(x) = \overline{a}\), we have \(a^*(y) > a^*(x)\). An entirely analogous argument establishes that \(a^*(y) < a^*(x)\) if \(x^\dagger < x < y\) and \(a^*(y) \neq \overline{a}\). Assertion (x) follows.

We are left with the task of establishing Assertion (viii). Given scalars \(\Delta > 0\) and \(x > x^\dagger + \Delta\), we define two processes
\[
x^-_t = x + \int_{s=0}^{t} (a^*(x^-_s) - \mu) dt + \sigma w_t,
\]
and
\[
x^+_t = x + 2\Delta + \int_{s=0}^{t} (a^*(x^+_s) - \mu) dt + \sigma w_t,
\]
each evolving on \([0, T]\), where \(T\) is given by
\[
T = \inf\{t|x^-_t = x^\dagger \text{ or } x^-_t = x^+_t\}.
\]
Let
\[
x_t = x + \Delta + \int_{s=0}^{t} \left( (a^*(x^+_s) + a^*(x^-_s)) / 2 - \mu \right) dt + \sigma w_t,
\]
and note that \(x_t = (x^+_t + x^-_t) / 2\) for all \(t \in [0, T]\). It is easy to show that \(T\) is finite with probability one.

Define “sample costs” associated with the three processes:
\[
\hat{J}(x, \omega) = \int_{t=0}^{T} e^{-\alpha t} \left( u(x_t) - c((a^*(x^+_t) + a^*(x^-_t)) / 2) \right) dt + e^{-\alpha T} J(x_T),
\]
\[
\hat{J}^+(x, \omega) = \int_{t=0}^{T} e^{-\alpha t} \left( u(x^+_t) - c(a^*(x^+_t)) \right) dt + e^{-\alpha T} J(x^+_T),
\]
\[
\hat{J}^-(x, \omega) = \int_{t=0}^{T} e^{-\alpha t} \left( u(x^-_t) - c(a^*(x^-_t)) \right) dt + e^{-\alpha T} J(x^-_T),
\]
where \(\omega\) denotes the sample path of the underlying Brownian motion \(w_t\).

We will show that for almost all \(\omega\) and any \(x \in (x^\dagger, \infty)\),
\[
\hat{J}(x, \omega) \geq \frac{1}{2} \left( \hat{J}^+(x, \omega) + \hat{J}^-(x, \omega) \right).
\]

We consider two separate cases that together comprise a set of probability 1. The first is when
In this event, we have \( x_T^- = x_T^+ = x_T > x^\dagger \), and the desired inequality follows directly from concavity of \( u \) and convexity of \( c \).

The second case is when \( x_T^- = x^\dagger \). Given our assumptions on \( c \), the fact that \( a^* \) is bounded above, and the fact that \( J \) is bounded and twice continuously differentiable on \((\underline{x}, \infty)\), it can be shown that for any \( y > \underline{x}, |a^*(y) - a^*(y+\Delta)| = O(\Delta) \). It follows that sup_{t\in[0,T]} |a^*(x_T^-) - a^*(x_T^+)| = O(\Delta) and \( x_T^+ - x^\dagger = O(\Delta) \). We then have

\[
\dot{J}(x, \omega) = \int_{t=0}^{T} e^{-\alpha t} \left( u(x_t) - c((a^*(x_T^+) + a^*(x_T^-))/2) \right) dt + e^{-\alpha T} J(x_T^-)
\]

where the second–to–last expression relies on the fact that \( J''(x^\dagger) = 0 \) and that \( x_T^+ - x^\dagger = O(\Delta) \).

It follows that for almost all \( \omega \) and any \( x \in (x^\dagger, \infty) \), \( \hat{J}(x, \omega) \) is concave in \( x \).

By Bellman’s principle of optimality, we have

\[
J(x) = E[\hat{J}^-(x, \omega)], \quad J(x + 2\Delta) = E[\hat{J}^+(x, \omega)], \quad \text{and} \quad J(x + \Delta) \geq E[\hat{J}(x, \omega)].
\]

Hence,

\[
J(x + \Delta) \geq \frac{1}{2} \left( J(x) + J(x + 2\Delta) + o(\Delta^2) \right),
\]

and therefore \( J''(x) < 0 \) for \( x > x^\dagger \).

q.e.d.

Next, consider Theorem 2, which contrasts the full SFQ case with the case where only abatement is possible. Before proving the theorem, we establish some properties of the optimal abatement policy in the abate-only case.

**Lemma 7** Let Assumptions 1 and 2 hold, and assume that restoration is not feasible. Let \( \bar{a} \) denote the optimal abatement policy in this case. Then (i) there exists a state \( \hat{x} \) such that \( \bar{a} \) is decreasing on \((\hat{x}, \infty)\) and \( \bar{a}(x) = \pi \) for all \( x \leq \hat{x} \); (ii) \( \lim_{x \to \infty} \bar{a}(x) = 0 \); and (iii) if \( \bar{a} > \mu \), then there exists a state \( x^*_{\text{abate}} \) such that \( \mu < \bar{a}(x) \) for \( x < x^*_{\text{abate}} \) and \( \mu > \bar{a}(x) \) for \( x > x^*_{\text{abate}} \).
Proof. Let \( \tilde{J} \) denote the optimal value function in the case when only abatement is possible:

\[
\tilde{J}(x) = \sup_a E_x^a \left[ \int_{t=0}^{\infty} e^{-\alpha t} (u(x_t) - c(a(x_t))) dt \right],
\]

where the supremum is taken over abatement policies. Let \( \tilde{a} \) be the corresponding optimal abatement policy.

Let \( \tilde{f}_x \) be defined by

\[
\tilde{f}_x(a) = a \tilde{J}_0(x) - c(a),
\]

and let \( \bar{a}(x) \) be the value in \([0, \pi]\) that uniquely attains the supremum of \( \tilde{f}_x \). Along similar lines as in the proof of Theorem 1, one can show that \( \tilde{J}_0(x) > \tilde{f}_0(x) > 0 \), implying \( \tilde{f}_x(0) > 0 \). Also recall that \( \tilde{f}_x'' = -c'' \le -\epsilon \) for some \( \epsilon \). Consider the less constrained problem

\[
\sup_{z \in [0, \infty)} \tilde{f}_x(z).
\]

Since \( \tilde{f}_x'' \le -\epsilon \), the supremum is always attained by some \( z \in (0, \infty) \). Let \( b(x) \) denote the optimum for a given state \( x \). Because \( \tilde{f}_x'(0) > 0 \), \( b(x) > 0 \). Furthermore, since \( \tilde{f}_x(z) \) decreases as \( x \) increases, \( b \) is decreasing.

It is easy to see that \( \bar{a}(x) = \min(b(x), \pi) \). Since \( \tilde{J} \) is unbounded below, for any \( z > 0 \) there exists a state \( x \) such that \( \tilde{f}_x(z) > 0 \), implying that \( b \) is unbounded above, and therefore, there exists a state \( \hat{x} \) such that \( \bar{a}(x) = \pi \) for \( x \le \hat{x} \). Assertion (i) follows.

Recall that \( \tilde{J} < J \) and \( \tilde{J}' > J' \), so that \( \lim_{x \to \infty} \tilde{J}'(x) = 0 \). Hence for any \( z > 0 \), there exists a state \( x \) such that \( \tilde{f}_x'(z) < 0 \), implying that \( \lim_{x \to \infty} b(x) = 0 \) and that Assertion (ii) holds. The fact that \( b \) is decreasing implies that there exists a state \( x^* \) such that \( \mu < b(x) \) for \( x < x^* \) and \( \mu > b(x) \) for \( x > x^* \). Since \( \mu < \pi \) by hypothesis, we have Assertion (iii).

q.e.d.

Proof of Theorem 2

Proof. As a step toward establishing Assertion (i), we will show that \( \tilde{J} < J \). It is easy to see that \( \tilde{J}' \le J' \). From Theorem 1, we have \( J'(\underline{x}) = 0 < \tilde{J}'(\underline{x}) \). This implies that \( \tilde{J}(\underline{x}) < J(\underline{x}) \). For \( x < \underline{x} \), we then have \( J(x) = J(\underline{x}) > \tilde{J}(\underline{x}) > \tilde{J}(x) \). For \( x > \underline{x} \), on the other hand, the fact that \( \tilde{J}(\underline{x}) < J(\underline{x}) \) follows from our observation that \( \tilde{J}(\underline{x}) < J(\underline{x}) \) coupled with standard sample-path arguments.

Consider two states \( y \) and \( z \) with \( \underline{x} \le \underline{y} < \underline{z} \). By Bellman’s principal of optimality (see, e.g.,
Krylov), we have

\[ \tilde{J}(z) = \sup_a E_z \left[ \int_{t=0}^{T} e^{-\alpha t} (u(x_t) - c(a(x_t))) dt + e^{-\alpha T} \bar{J}(y) \right] \]

and

\[ J(z) = \sup_{a,R} E_{z,R} \left[ \int_{t=0}^{T} e^{-\alpha t} (u(x_t) - c(a(x_t))) dt + e^{-\alpha T} \bar{J}(y) \right] \]

\[ = \sup_a E_z \left[ \int_{t=0}^{T} e^{-\alpha t} (u(x_t) - c(a(x_t))) dt + e^{-\alpha T} \bar{J}(y) \right] , \]

where \( T \) is the first time at which \( x_t = y \). (The final equality holds because \( x_t > x \) for \( t \leq T \).

Let \( \bar{a} \) be an optimal policy for the case where only abatement is possible. We then have

\[ J(z) - \tilde{J}(z) = \sup_a E_z \left[ \int_{t=0}^{T} e^{-\alpha t} (u(x_t) - c(a(x_t))) dt + e^{-\alpha T} \bar{J}(y) \right] \]

\[ - \sup_a E_z \left[ \int_{t=0}^{T} e^{-\alpha t} (u(x_t) - c(a(x_t))) dt + e^{-\alpha T} \bar{J}(y) \right] \]

\[ \leq E_z^{a^*} \left[ \int_{t=0}^{T} e^{-\alpha t} (u(x_t) - c(a^*(x_t))) dt + e^{-\alpha T} \bar{J}(y) \right] \]

\[ - E_z^{a^*} \left[ \int_{t=0}^{T} e^{-\alpha t} (u(x_t) - c(a^*(x_t))) dt + e^{-\alpha T} \bar{J}(y) \right] \]

\[ = E_z^{a^*} \left[ e^{-\alpha T} (J(y) - \bar{J}(y)) \right] \]

\[ < J(y) - \bar{J}(y). \]

It follows that \( J' < \bar{J}' \), which gives us Assertion (i).

Now turn to Assertion (ii). Again, let \( \bar{f}_x \) be defined by \( \bar{f}_x(a) = a \tilde{J}(x) - c(a) \). Recall that for any \( x \), the supremum of \( \bar{f}_x \) is uniquely attained by \( \bar{a}(x) \). Since \( \tilde{J}' > J' \), for every \( x > \bar{x} \), we have \( \tilde{f}_x(a^*(x)) > f'_x(a^*(x)) \). This implies that if \( a^*(x) < \overline{\pi} \) then \( \bar{a}(x) > a^*(x) \). Hence, we have Assertion (ii).

q.e.d.

**Proof of Theorem 3**

**Proof.** Let \( J(x, C) \) denote the optimal value of state \( x \) given a restoration cost \( C > 0 \). It is easy to show by a sample path argument that for any \( x \), \( J(x, C) \) is decreasing in \( C \). Fix \( C_2 > C_1 > 0 \)
and assume for contradiction that $z(C_2) \geq z(C_1)$. Let $T = \inf \{ t | x_t = z(C_2) \}$. We then have

$$J(0, C_2) = E_0 \left[ \int_{t=0}^{T} e^{-\alpha t} u(x_t) dt + e^{-\alpha T} J(z(C_2), C_2) \right]$$

$$= E_0 \left[ \int_{t=0}^{T} e^{-\alpha t} u(x_t) dt + e^{-\alpha T} J(z(C_2), C_1) \right]$$

$$+ E_0 \left[ e^{-\alpha T} (J(z(C_2), C_2) - J(z(C_2), C_1)) \right]$$

$$= J(0, C_1) + E_0 \left[ e^{-\alpha T} (J(z(C_2), C_2) - J(z(C_2), C_1)) \right]$$

$$< J(0, C_1) + J(z(C_2), C_2) - J(z(C_2), C_1).$$

It follows that

$$J(0, C_2) - J(z(C_2), C_2) < J(0, C_1) - J(z(C_2), C_1) \leq J(0, C_1) - J(z(C_1), C_1),$$

(10)

where the final inequality relies on our assumption that $z(C_2) \geq z(C_1)$.

Theorem 1 asserts that for any $C > 0$, $J(0, C) - J(z(C), C) = C$. Inequality 10 therefore implies that $C_2 < C_1$, which yields a contradiction.

q.e.d.

Proof of Theorem 4

Proof. We have

$$Y_i^* = \int_{t=T_i-1}^{T_i} \left[ -\tau^*(x_t) dx_t + \rho^*(x_t) \sum_j \sigma_j^2 dt \right],$$

letting $T_0 \equiv 0$. Substituting for $\tau^*(x)$ and $\rho^*(x)$, and noting that $\sum_j \sigma_j^2 = \sigma^2$, we have

$$Y_i^* = - \int_{t=T_i-1}^{T_i} \left[ J'(x_t) dx_t + \frac{\sigma^2}{2} J''(x_t) dt \right] = C,$$

by equation (6).

q.e.d.

Proof of Theorem 5

Proof. We have

$$\bar{Y}_i = \int_{t=T_i-1}^{T_i} \left( \tau^*(x_t) (\mu - a_t) + \rho^*(x_t) \sum_j \sigma_j^2 \right) dt.$$
We first establish that $E[\bar{Y}_t] = E[Y_t^*]$. Note that

$$E \left[ - \int_{t=T_{i-1}}^{T_i} \tau^*(x_t) \, dx_t \right] = E \left[ - \int_{t=T_{i-1}}^{T_i} \tau^*(x_t) \left( (a_t - \mu) \, dx_t + \sigma \, dw_t \right) \right] = E \left[ - \int_{t=T_{i-1}}^{T_i} \tau^*(x_t) (a_t - \mu) \, dx_t \right].$$

The second line follows from the fact that $E \left[ \int_{s=T_{i-1}}^{T_i} \tau^*(x_s) \sigma \, dw_s \right] = 0$ by the optimal sampling theorem: note that $\int_{s=T_{i-1}}^{t} \tau^*(x_s) \sigma \, dw_s$, conditioned on $T_{i-1}$, is a martingale, and $T_i$ is a stopping time. Hence $E[Y_t^*] = E \left[ \int_{s=T_{i-1}}^{T_i} \left( -\tau^*(x_t) (a_t - \mu) + \rho^*(x_t) \sum_j \sigma_j^2 \right) \, dx_t \right] = E[\bar{Y}_t]$. But $E[Y_t^*] = E[C] = C$ by Theorem 4, thus $E[\bar{Y}_t] = C$ as required.

\textbf{q.e.d.}

\textbf{Proof of Theorem 6}

\textbf{Proof.} Without loss of generality, we will assume in this proof that $\sigma \geq 0$. Recall that damage evolves according to $z_t = \mu t - \sigma w_t$. Consider a fixed restoration threshold $\tilde{x} < 0$, which may or may not correspond to the optimal restoration strategy. We introduce some notation to facilitate our analysis. First, we denote the running maximum of damage by $\bar{r}(t)$, together form an ergodic process. There is a joint stationary distribution over the variables $r_t$.

Let $J_\bar{x}(\cdot, \sigma, \alpha)$ be the value function corresponding to a restoration threshold $\tilde{x}$. Since $x_t$ reaches $\tilde{x}$ in finite expected time and the process regenerates every time it hits $\tilde{x}$, it is ergodic. It follows that

$$\lim_{\alpha \downarrow 0} \alpha J_\bar{x}(x, \sigma, \alpha) = \lim_{\alpha \downarrow 0} \alpha E \left[ \int_{t=0}^{\infty} e^{-\alpha t} u(x_t) \, dt + \sum_{i=1}^{\infty} e^{-\alpha \tau_i} C \right] = \lim_{T \to \infty} \frac{1}{T} E \left[ \int_{t=0}^{T} u(x_t) \, dt + r_T C \right] = \lim_{T \to \infty} \frac{1}{T} E \left[ \int_{t=0}^{T} u(x_t) \, dt \right] - C \mu \tilde{x},$$

where the final term follows from the fact that the expected interarrival time between visits to $\tilde{x}$ is $-\mu \tilde{x}$.

We will now establish that $\lim_{\alpha \downarrow 0} \alpha J_\bar{x}(x, \sigma, \alpha)$ is increasing in $\sigma$. Note that $(x_t, y_t, r_t - r\tilde{r}_\lambda)$ together form an ergodic process. There is a joint stationary distribution over the variables $x_t$, $y_t$, $r_t$.
$y_t$, and $r_t - r_0$ such that if $(x_0, y_0, r_0 - r_0)$ is sampled from this distribution, $(x_t, y_t, r_t - r_0)$ is a stationary process. Let $E_\infty$ denote expectation with respect to the distribution of this stationary process. It is easy to see that, for any $t$, the marginal distribution (with respect to the stationary process) of $y_t$ is uniform over $[\tilde{x}, 0]$. We therefore have

$$
\lim_{T \to \infty} \frac{1}{T} E_x \left[ \int_{t=0}^{T} u(x_t) dt \right] = \lim_{T \to \infty} \frac{1}{T} E_\infty \left[ \int_{t=0}^{T} u(x_t) dt \right] = \frac{1}{T} E_\infty \left[ \int_{t=0}^{T} u(x_t) dt \right] = \frac{1}{T} \int_{y=0}^{\tilde{x}} E_\infty \left[ \int_{t=0}^{T} u(y_t - (r_t - r_0)\tilde{x}) dt \mid y_t = y \right] dy.
$$

Note that, conditioned on $z_0$ and $z_t$, the process $z_\tau$ forms a Brownian bridge on $\tau \in [0, t]$. A sample path argument shows that for any $\gamma > \max(z_0, z_t)$, $\Pr\{m_t \geq \gamma \mid z_0, z_t\}$ is increasing in $\sigma$. It follows that for any $\gamma > \max(z_0, z_t)$, $\Pr\{m_t - z_t \geq \gamma \mid z_0, z_t\}$ is increasing in $\sigma$, and therefore, for any $\gamma \geq 1$, $\Pr\{r_t - r_0 \geq \gamma \mid z_0, z_t\}$ is increasing in $\sigma$. Since this holds for all $z_0$ and $z_t$, and $y_t$ is a deterministic function of $z_t$, for any $\gamma \geq 1$ and any $y_t$, $\Pr\{r_t - r_0 \geq \gamma \mid y_t\}$ is also increasing in $\sigma$. Since $u' > 0$, it follows that

$$
E_\infty \left[ \int_{t=0}^{T} u(y_t - (r_t - r_0)\tilde{x}) dt \mid y_t \in dy \right] dy
$$

is increasing in $\sigma$. Therefore,

$$
\lim_{T \to \infty} \frac{1}{T} E_x \left[ \int_{t=0}^{T} u(x_t) dt \right]
$$

is increasing in $\sigma$. It follows that $\lim_{\alpha \to 0} \alpha J_{\tilde{x}}(x, \sigma, \alpha)$ is increasing in $\sigma$.

It is not hard to show that for any $x > \tilde{x}$ and any $\alpha > 0$, $J_{\tilde{x}}(x, \sigma, \alpha)$ is continuously differentiable in $\tilde{x}$ and $\sigma$, and we will take this as given. Let $\tilde{x}(\sigma, \alpha)$ denote the optimal threshold as a function of $\sigma$ and $\alpha$. It can be shown that $\tilde{x}(\sigma, \alpha)$ is continuously differentiable in $\sigma$, and we take this as given as well. It follows that

$$
\frac{\partial J_{\text{restore}}(x, \sigma, \alpha)}{\partial \sigma} \bigg|_{\sigma = \sigma'} \frac{\partial J_{\tilde{x}}(x, \tilde{x}, \alpha)}{\partial \tilde{x}} \bigg|_{\tilde{x} = x(\sigma, \alpha)} \frac{\partial x(\sigma, \alpha)}{\partial \sigma} \bigg|_{\sigma = \sigma'} + \frac{\partial J_{\tilde{x}}(x, \sigma, \alpha)}{\partial \sigma} \bigg|_{\sigma = \sigma'}.
$$

Since $x(\sigma, \alpha)$ maximizes $J_{\tilde{x}}(x, \sigma, \alpha)$ over $\tilde{x} \in \mathbb{R}$, we have

$$
\frac{\partial J_{\tilde{x}}(x, \sigma, \alpha)}{\partial \tilde{x}} \bigg|_{\tilde{x} = x(\sigma, \alpha)} = 0.
$$

We have already shown that, for any $x$ and $\sigma > 0$, $\lim_{\alpha \to 0} \alpha J_{\tilde{x}}(x, \sigma, \alpha)$ is increasing in $\sigma$. It follows

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that, for any $x$ and $\sigma > 0$, there exists some $\alpha > 0$ such that for all $\alpha \in (0, \overline{\alpha})$,

$$\frac{\partial J_{\text{z}(\sigma, \alpha)}(x, \sigma, \alpha)}{\partial \sigma} \bigg|_{\sigma = \overline{\alpha}} > 0,$$

and therefore

$$\frac{\partial J_{\text{restore}}(x, \sigma, \alpha)}{\partial \sigma} \bigg|_{\sigma = \overline{\alpha}} > 0.$$

q.e.d.

Appendix B: Numerical simulations in Section 3

The computations that generated the figures were conducted using a quadratic function for abatement cost and a negative natural exponential function for utility. The functional forms and parameter values used are summarized in Table B1. The flow rate $\mu$ is not given in the table: it varied as indicated in the figures and the text. The variance rate $\sigma^2$, the marginal cost parameter $\gamma$, and the discount rate $\alpha$ also vary in some figures, as indicated.

Value functions were computed via policy iteration on a “locally consistent” approximating Markov chain (see, e.g., Kushner and Dupuis, 1992). Most simulations required only 10 iterations to converge to a solution, although more iterations were used in some cases.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>variance rate</td>
<td>$\sigma^2 = 9.0$</td>
</tr>
<tr>
<td>discount rate</td>
<td>$\alpha = 0.005$</td>
</tr>
<tr>
<td>restoration cost</td>
<td>$C = 13000$</td>
</tr>
<tr>
<td>abatement ceiling</td>
<td>$\overline{\sigma} = 20$</td>
</tr>
<tr>
<td>abatement cost</td>
<td>$c(a) = \gamma a^2$</td>
</tr>
<tr>
<td>utility</td>
<td>$u(x) = -e^{-\beta x + \kappa}$</td>
</tr>
<tr>
<td></td>
<td>$\beta = 0.05$</td>
</tr>
<tr>
<td></td>
<td>$\kappa = -7.5$</td>
</tr>
</tbody>
</table>
Figure 1: Optimal abatement path and corresponding value function. Note the different units of measurement on the positive and negative segments of the vertical axis.

Figure 2: Value functions with and without the availability of abatement.
Figure 3: Effects of restoration on optimal abatement policy, for two flow rates.
Figure 4: Optimal restoration trigger as a function of the mean flow rate and variability. In the bottom panel, variability $\sigma^2$ and flow rate $\mu$ are depicted along the $x$ and $y$ axes, respectively, with $\sigma^2$ ranging over (1, 10, 20, ..., 100) and $\mu$ ranging from 1 to 5 in increments of 0.5. Hence the four sides of surface depicted in the figure run along the sides of the enclosing “box.” The top two panels show the two-dimensional projections of the same surface.
Figure 5: Optimal abatement as a function of quality, for three flow rates. The left-hand figure (panel (a)) plots abatement rates; the right-hand figure (panel (b)) depicts the abatement rates as fractions of mean flow rates.

Figure 6: The effect of the variance rate $\sigma^2$ on the optimal abatement policy.
Figure 7: Fraction of optimal melioration due to abatement, as a function of the mean flow rate $\mu$.

Figure 8: Frequency distributions of resource qualities (states) under optimal policies for three flow rates.

Figure 9: The effect of the discount rate $\alpha$ on the optimal abatement policy.