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Ross Baldick and William Hogan

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University of California Energy Institute
2539 Channing Way
Berkeley, California 94720-5180
www.ucei.org

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Ross Baldick
Department of Electrical and Computer Engineering
The University of Texas at Austin

William Hogan
Center for Business and Government
John F. Kennedy School of Government
Harvard University

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Abstract

In this paper we consider a supply function model of an electricity market where strategic firms have capacity constraints. We show that if firms have heterogeneous cost functions and capacity constraints then the differential equation approach to finding the equilibrium supply function may not be effective by itself because it produces supply functions that fail to be non-decreasing. Even when the differential equation approach yields solutions that satisfy the non-decreasing constraints, many of the equilibria are unstable, restricting the range of the equilibria that are likely to be observed in practice. We analyze the non-decreasing constraints and characterize piece-wise continuously differentiable equilibria. To find stable equilibria, we numerically solve for the equilibrium by iterating in the function space of allowable supply functions. Using a numerical example based on supply in the England and Wales market in 1999, we investigate the potential for multiple equilibria and the interaction of capacity constraints, price caps, and the length of the time horizon over which bids must remain unchanged. We empirically confirm that the range of stable supply function equilibria can be very small when there are binding price caps. Even when price caps are not binding, the range of stable equilibria is relatively small. We find that requiring supply functions to remain fixed over an extended time horizon having a large variation in demand reduces the incentive to mark up prices compared to the Cournot outcome.

1 Introduction

Supply function equilibrium models were developed by Klemperer and Meyer in [1] to analyze markets where agents bid a schedule of price-quantities. The original motivation was to handle random shocks in demand that could be characterized by a continuous random variable having convex support. Their approach sets up coupled differential equations that, under certain circumstances, characterize the equilibrium.

In recent papers, supply function equilibrium models have been applied to analysis of electricity markets [2, 3, 4, 5, 6, 7, 8, 9]. This approach, pioneered by Green and Newbery [2], reinterprets the probability distribution of random shocks in [1] to be an electricity load-duration characteristic. The support of the probability distribution becomes the range of demands in the load-duration characteristic.

Unfortunately, without restrictive assumptions on the nature of the costs and capacity constraints, on the number of firms, or on the form of the allowed bid functions, it has proven difficult to find equilibria in supply functions. For example, in [8], to obtain a convenient characterization of the equilibrium, the authors assume that each bidder must submit either:

- an affine supply function or
- a piece-wise affine supply function where the number of pieces is relatively small.

In the case of minimum capacity constraints, [8] exhibits a piece-wise affine supply function equilibrium. Representing maximum capacity constraints prompts an *ad hoc* approach in [8] that attempts to approximate the equilibrium supply functions when there are maximum capacity constraints.

In this paper, we relax the assumption of [8] that the bidders submit a supply function consisting of a small number of pieces. We analyze the properties of the equilibrium and also numerically estimate candidate equilibrium supply functions by iterating in the function space of allowable bids; however, in practice this means that we still must approximate the supply functions with a piece-wise affine and continuous function, albeit having a large number of pieces.

We qualify the numerical estimates of the equilibria as “candidate” because the functions we calculate cannot be guaranteed to be equilibria without a further check of global optimality of each bidder's bid (given everyone else's bid.) We do not perform this even more computationally intensive calculation. However, as argued in [9], the “limited optimizing behavior” that we simulate may nevertheless be a reasonable model for gaining some insight about plausible bidder behavior.

We investigate one basic criticism of supply function equilibrium analysis: that there are multiple supply function equilibria so that the approach has little predictive value. As a first response to this criticism, Green and Newbery [2, § II.B] note that capacity constraints tend to limit the range of equilibria. They describe conditions for uniqueness in an extreme case where the capacity constraints are so tight that the price at peak demand in the supply function equilibrium is as high as the price under Cournot competition.

We find that although there is a continuum of equilibria in the uncapacitated case, the range of equilibria is less likely to be problematic when there are moderately tight capacity constraints and price caps. This echos the observations by Green and Newbery but goes further in that we find that the presence of price caps yields unique equilibria with prices well below the Cournot price. Moreover, we show that even when there is a wide range of equilibria, many of these equilibria are unstable and so are unlikely to be observed in practice. Our analysis confirms a suggestion made in [10] that "an equilibrium is less likely to be stable if it involves generators offering power at prices very much higher than their marginal costs" [10, page 20].

We then use the numerical calculations to explore the interaction of three issues:

1. the effect of price caps (set above the maximum marginal cost of production) in an institutional framework where firms are obliged to supply all their available generation capacity whenever the price reaches the price cap,
2. the effect of maximum capacity constraints on strategic behavior, and
3. the effect of requiring that supply function bids be fixed over an extended time horizon during which demand varies essentially continuously.

We discuss these issues in the following paragraphs.

The assumption that bidders must sell all their capacity at the price cap does not accurately represent those markets with price caps where either:

- the bidders have alternate sales opportunities that are not price-capped or
- the bidders can otherwise declare their capacity to be unavailable to the market.

However, the assumption should provide a lower bound on the amount of capacity withholding that might occur in a real market. Our assumption is intended to reflect the intent of regulatory authorities in setting price caps: presumably they expect that all capacity will be offered whenever the price reaches the price cap.

We also consider the alternative of a *bid cap,* where there is a limit on the bid prices but the market prices can rise above this level to limit the demand. Bid caps have been proposed as a means to limit market power when there are transmission constraints, while also allowing prices to rise to high levels to reflect the true cost of a constraint. We investigate their application in transmission unconstrained systems where the generation capacity is limited.

Generation maximum capacity constraints are pervasive in electricity markets. As discussed in [8], the presence of capacity constraints complicates the determination of conditions for profit maximization because the profit functions are typically non-concave. We will discuss this issue in the context of a profit function defined over a time horizon.

Some markets, such as the England and Wales market until 2001, explicitly require bidders to submit a single supply function valid (essentially) for a whole day. Requiring supply function bids to be fixed over an extended time horizon means that bidders must balance the desire to withhold capacity when prices are high against sales opportunities at lower prices. In contrast, other markets, such as the (now defunct) California Power Exchange, allow different bid functions every hour.

Issues such as start-up costs, ramp rate limits, and environmental constraints couple generation costs from hour to hour. Moreover, capacity can change due to outages. However, the production cost function of an in-service generator may not change significantly on an hour by hour basis, so that the flexibility to bid different supply functions on an hour by hour basis is not obviously justified by technical issues, except to the extent that start-up costs, ramp rate limits, environmental constraints, and changing fuel costs are significant.

We consider the incentives due to requiring consistent bids over an extended time horizon; however, we do not consider how to handle start-up costs, ramp rates and environmental constraints nor the institutional oversight required to enforce bid consistency [2, § II.B]. There are admitted difficulties in trying to enforce consistency of bids. For example, in the England and Wales market, although bids were fixed over a day, declared capacities could be changed, effectively redefining the bid. Also, day-ahead markets typically have hourly or real-time markets. Implicit in our analysis is the assumption that most volume is traded in the day-ahead market.

We assume that the load-duration characteristic is continuous over the time horizon. This is analogous to the Klemperer and Meyer assumption that the random variable representing the demand shock has convex support [1].

To investigate the three issues of price caps, maximum capacity constraints, and the requirement to bid supply functions that are consistent over an extended time horizon, we perform numerical calculations using cost data that are based on that in [8] for the five strategic firm industry in England and Wales subsequent to the 1999 divestiture. Our demand and price cap assumptions are, however, fictitious and simply chosen to highlight the effects of capacity constraints, price caps, and an extended time horizon. Naturally, caution should be exercised in extrapolating the numerical results to other cases.

We assume that all energy is sold at the marginal clearing price. More recently, the England and Wales market has changed to a pay-as-bid structure; however, we have not modeled this new market structure.

The main findings of this work are:

- In markets with firms having heterogeneous cost functions and capacity constraints, the differential equation approach to finding the equilibrium supply function may not be effective by itself.
- The range of supply function equilibria can be very small when there are binding price caps. Even when price caps are not binding, the range of stable equilibria appears small compared to the difference between, say, the competitive and the Cournot outcomes.
- Requiring supply functions to remain fixed over an extended time horizon having a large and continuous variation in demand appears to reduce the incentive to mark up prices compared to the Cournot outcome.
- A single price cap imposed at all times may have significant effects both on- and off-peak.

The third observation is consistent with the results in [11], which used an "adaptive agent" approach to evaluate the incentives of daily and hourly bidding in the England and Wales market.

The outline of the paper is as follows. The formulation is described in section 2, with the assumptions and formulation essentially standard from the supply function equilibrium literature. Section 3 then explores the approach to solving the equilibrium conditions as a coupled differential equation. In section 4 we discuss some of the assumptions of the model in detail, highlighting three issues that are critical in the analysis of section 3:

1. consistency of bids across the time horizon,
2. continuity of the load-duration characteristic, and
3. the nature of the marginal cost functions.

We use a three firm example based on an example in [9] to illustrate the effect of requiring consistency of bids across the time horizon on the range of equilibria.

We next consider stability. There are various time scales in the operation of an electric power system, from sub-second to longer than a day. At the sub-second time scale, the electromechanical interactions must be analyzed for stability. At a slightly slower time scale, short-term electric power markets have dynamics that can potentially interact with the electromechanical dynamics. Alvarado *et al.* analyze these interactions [12]. Our interest is in the stability of the economic equilibria. Alvarado considers electricity market stability in a quantity bidding context [13]. Anderson and Xu considers stability of supply function equilibria in [10].

In section 5, we analyze the stability of the supply function equilibria calculated using the differential equation approach and present a theorem that characterizes unstable supply function equilibria. This theorem sheds light on why the apparent multiplicity of supply function equilibria may not be as serious a problem as implied by the apparently wide range of possible solutions of the differential equations. We again use the three firm example to illustrate how the stability analysis restricts the range of equilibria that are likely to be observed in practice.

In section 6 we then present a theorem that suggests why the coupled differential equation approach is not likely to be fruitful in the case of firms having capacity constraints and asymmetric cost functions. The reason is that the solutions of the differential equations will not usually satisfy the requirement that the supply functions be non-decreasing across the range of realized prices. We illustrate this theorem with a five firm example system based on the England and Wales system [8].

We complement the analysis in section 6 with a further analysis of the non-decreasing constraints in section 7. This analysis provides a characterization of the properties of piecewise continuously differentiable SFEs. In particular, we show that while the range of the load-duration characteristic affects the set of possible supply function equilibria, the set of possible equilibria is not affected by the detailed functional form of the load-duration characteristic.

We use a two firm example system to illustrate an apparently paradoxical property of supply function equilibria. In particular, the non-decreasing constraints are not apparently binding on the equilibrium solutions in the sense that the equilibrium solutions are typically all strictly increasing. However, these constraints are actually binding in the sense that if the non-decreasing constraints were relaxed for a particular firm then its optimal response would be different. This apparent paradox is due to the fact that the profit function for a firm can be non-concave, so that apparently non-binding constraints actually cut off solutions that have higher profit than the feasible solutions.

Section 8 describes an approach to finding the SFEs that involves iterating in the function space of supply functions. Section 10 discusses the detailed assumptions in the numerical implementation, while case studies and results are presented in section 11 based on the five firm example system. The case studies first investigate numerically the issue of multiplicity of equilibrium solutions. Then the effect of varying price caps, capacities, the load factor, and demand are investigated. We conclude in section 12.

2 Formulation

In this section, we first discuss the demand, generation costs and capacities, and supply functions. Then we discuss price and price caps, assumptions on the form of the supply functions, the profit, and the equilibrium conditions. The development is standard.

2.1 Demand

Following Green [3], we assume that *demand* $D : \mathbb{R}_+ \times [0,1] \rightarrow \mathbb{R}$ is a continuous function of the form:

$$\forall p \in \mathbb{R}_+, \forall t \in [0,1], D(p,t) = N(t) - \gamma p, \tag{1}$$

where:

- p is the price,
- t is the (normalized) time,
- $N : [0,1] \rightarrow \mathbb{R}_+$ is the *load-duration* characteristic, and
- $\gamma \in \mathbb{R}_+$ is minus the slope of the demand curve.

That is, the demand is assumed to be additively separable in its dependence on price and on time. The load-duration characteristic N represents the distribution of demand over a time horizon, with:

- the time argument t normalized so that it ranges from 0 to 1 and
- N non-increasing, so that $t = 0$ corresponds to peak conditions and $t = 1$ corresponds to minimum demand conditions.

Figure 1 illustrates a linear load-duration characteristic.

The assumption of a linear demand-price dependence and a linear load-duration characteristic is somewhat restrictive. More complicated continuous load-duration characteristics D can easily be accommodated in the computational model we develop; however, as we will see, the functional form of the load-duration characteristic does not affect the set of equilibria. Other demand-price dependencies such as constant elasticity could also be represented, but this would require more substantial modifications.

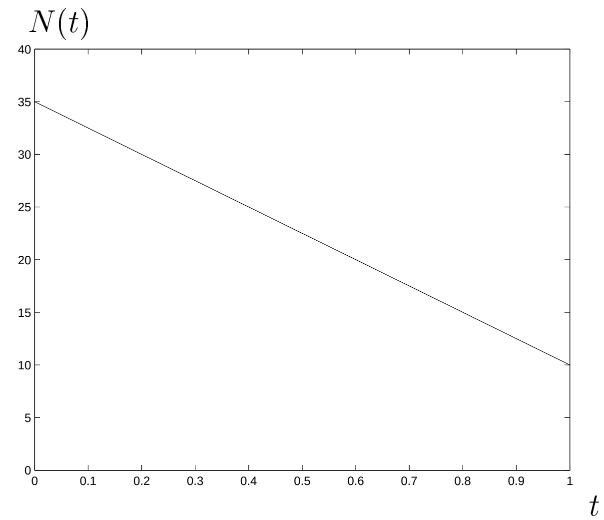

Figure 1: Example load-duration characteristic.

2.2 Generation costs and capacities

We assume that firms are labeled $i = 1, \ldots, n$, with $n \geq 2$. Following [8] and except as noted, we will assume that the *total variable generation cost function* $C_i : \mathbb{R}_+ \to \mathbb{R}$ of the i-th firm is quadratic and of the form:

$$\forall q_i \in \mathbb{R}_+, C_i(q_i) = \frac{1}{2} c_i q_i^2 + a_i q_i,$$

with $c_i \geq 0$ for each i so that the variable generation costs are convex. We therefore ignore issues such as start-up and minimum-load costs. We use superscript $'$ to represent differentiation and denote the marginal cost by C_i', so that:

$$\forall q_i \in \mathbb{R}_+, C_i'(q_i) = c_i q_i + a_i. \tag{2}$$

Each firm is assumed to be able to produce down to zero output, so that the minimum capacity constraints are all equal to zero. Each firm has a maximum capacity $\overline{q}_i$. That is, the *capacity constraints* for the firms require that:

$$\forall i, 0 \leq q_i \leq \overline{q}_i. \tag{3}$$

The cost function C_i represents the variable generation cost function of the whole firm i. Typical firms own several generation units, including several technologies such as coal, oil, and natural gas. Moreover, typical generation units have increasing marginal costs over their operating range of production. Therefore, C_i can be construed as resulting from optimal economic dispatch of the portfolio of generation owned by firm i.

The assumption of affine marginal costs does not capture jumps in marginal cost from, say, coal to gas technology and does not capture the rapid increase in marginal costs at high output close to the maximum capacity. However, it does represent the qualitative observation of increasing marginal cost with output. That is, we will usually have that $c_i > 0$.

More complicated marginal cost curves could easily be incorporated into the computational model. For example, a "barrier term" could be added to the cost function to represent a rapid rise in marginal costs as q_{it} approaches $\overline{q}_i$.

2.3 Supply functions

As discussed in the introduction, in the formulation of Klemperer and Meyer [1] a probability distribution characterizes a range of random demand outcomes. Bushnell and Wolak [14] use such a model to investigate optimal hourly responses in the California electricity market.

In contrast, Green and Newbery [2] and Green [3] model deterministic variation of demand over an extended time horizon. We follow this approach, assuming that each firm bids a *supply function* into the market; that is, a function $S_i : \mathbb{R}_+ \to \mathbb{R}$ that represents the amount of power it is willing to produce at each specified price per unit energy. (We will restrict the functional form of the S_i further in section 2.5 and definition 1.) The supply function applies throughout the time horizon specified by the load-duration characteristic. For example, in the England and Wales until 2001, a new supply function could be specified for each day so that the load-duration characteristic could be considered to be of one day duration.

Analysis of hybrid situations is also possible, where D represents the distribution of demand over a day but the demand is not completely deterministic. In such a hybrid case, S_i still applies throughout the time horizon and responds both to the variation of demand over the time horizon and also to the uncertainty of demand at each time.

We investigate how the load factor over the time horizon affects the equilibrium outcomes. We will observe that requiring bids to be consistent over extended time horizons that include the peak conditions and also lower demand conditions can have a significant effect on limiting price mark ups and equilibrium profits. However, we recognize that such non-cooperative equilibrium analyses also understate the level of market power because they neglect the possibility of collusion and the impact of repeated interactions.

2.4 Price cap and price minimum

Price caps are in place in many electricity markets. The detailed implementation of the price caps varies from market to market. To represent the effect of a generic price cap on the market, we follow von der Fehr and Harbord [15] and assume that the market rules specify a price cap $\overline{p}$ and that the firms are obliged to bid supply functions that satisfy:

$$\forall i, S_i(\overline{p}) = \overline{q}_i. \tag{4}$$

That is, each firm must be willing to operate at full output if the price reaches the price cap. Of course, firms might also bid so that they would be prepared to produce at full output for lower prices.

As discussed in [16, § V], enforcement of this requirement necessitates that the market operator be prepared to curtail demand and not breach the price cap. Furthermore, the market operator must be able to reliably estimate the maximum marginal cost of production by any firm in the market so that the price cap can be set above the maximum marginal cost of production.

We assume for convenience that there is a known minimum price $\underline{p}$ below which no firm would be prepared to bid any non-zero supply. For example, $\underline{p} = \min_i\{a_i\}$ is a suitable value since no firms will be willing to generate for a price that falls below the marginal operating costs at zero output of the cheapest generator.

2.5 Feasible and allowable supply functions

We require that each supply function be defined for every price in the interval $[\underline{p}, \overline{p}]$. To be *feasible* the range of the supply function for firm i must be contained in the interval $[0, \overline{q}_i]$. That is, the supply function for firm i is a function $S_i : [\underline{p}, \overline{p}] \to [0, \overline{q}_i]$.

Market rules require that supply functions be *non-decreasing* in order to be *allowable* as bids. That is, $p \leq p' \Rightarrow S_i(p) \leq S_i(p')$. Some authors appear to neglect this constraint. For example, Bolle [17] presents supply function equilibrium solutions that fail to satisfy the non-decreasing constraints. (See [17, Figure 2].) We will find that the non-decreasing constraints must be represented in the model. (However, we will also observe that the non-decreasing constraints are not apparently binding at the equilibrium.)

The requirement that each supply function be feasible and allowable is embodied in the following:

Definition 1 For each $i = 1, \ldots, n$, the set $\mathbb{S}_i$ is the function space of feasible and allowable supply functions for firm i having domain $[\underline{p}, \overline{p}]$. That is, $\mathbb{S}_i$ is the set of functions with domain $[\underline{p}, \overline{p}]$ that:

1. have range $[0, \overline{q}_i]$ (so that all bids are feasible for allowed prices) and
2. are non-decreasing over the domain $[\underline{p}, \overline{p}]$, (so that the function is an allowable supply function).

□

In section 3 when we analyze differential equations with solutions that yield supply function equilibria, we will further restrict $\mathbb{S}_i$ to be the space of *differentiable* functions that are feasible and allowable. In this case, the non-decreasing constraints are equivalent to:

$$\forall i = 1, \ldots, n, \forall p \in [\underline{p}, \overline{p}], S_i'(p) \geq 0,$$

where superscript $'$ denotes differentiation.

2.6 Price

At each time $t \in [0, 1]$, the market is cleared based on the bid supply functions $S = (S_i)_{i=1,\ldots,n}$ and the demand. That is, at each time t, the price is determined by the solution of:

$$D(t, p) = N(t) - \gamma p = \sum_i S_i(p), \tag{5}$$

assuming a solution exists. All firms receive the marginal clearing price for their supply. We say that this price corresponds to the bid supply functions S.

If $\gamma > 0$ then for each t and each collection of choices of non-decreasing supply functions $S_i \in \mathbb{S}_i, i = 1, \ldots, n$ there is at most one solution to (5) having $\underline{p} \leq p \leq \overline{p}$. If there is a solution to (5) in this range, then this solution determines the price at time t. (If S_i is discontinuous then we must modify the notion of "a solution to (5)" slightly; however, we will not need to deal with this issue for the supply functions we exhibit.) If there is no solution to (5) in the range $\underline{p} \leq p \leq \overline{p}$, then the realized price depends on whether the market is assumed to have price caps or bid caps. We discuss these two cases in the next sections.

2.6.1 Price caps

In the case of price caps, the market price is never allowed to rise above $\overline{p}$. If there is insufficient supply to meet the demand at price $p = \overline{p}$ then demand must be rationed. In this case, we will assume that:

- demand is rationed to the available supply and
- all energy is sold at a price equal to the price cap.

For any particular choices $S_i, i = 1, \ldots, n$, we can therefore implicitly solve for price as a function of time. That is, there is a function $P : [0, 1] \to [\underline{p}, \overline{p}]$, which is parameterized by $S_j \in \mathbb{S}_j, j = 1, \ldots, n$, such that:

$$\forall t \in [0, 1], D(t, P(t; S_j, j = 1, \ldots, n)) \geq \sum_i S_i(P(t; S_j, j = 1, \ldots, n)), \tag{6}$$

with equality between the left and right hand sides except at times when demand rationing occurs. For notational convenience, we will omit the explicit parameterization of the function P and just write it with one argument, namely, the normalized time t. Occasionally, we will need to consider price functions arising from alternative choices of supply functions. In this case, we will distinguish the price functions by superscripts. For example, in sections 5 and 7, we will consider supply functions $S_i^\epsilon, i = 1, \ldots, n$. We will denote the resulting price function P^ϵ.

2.6.2 Bid caps

In this alternative market structure, prices can rise to higher than $p = \overline{p}$ in order to ration demand based on price. That is, there is a cap on bids but not on prices. To implement the bid caps, we implicitly extrapolate the supply functions to being functions $S_i : [\underline{p}, \infty) \to [0, \overline{q}_i]$ by defining:

$$\forall i, \forall p > \overline{p}, S_i(p) = \overline{q}_i.$$

Moreover, we relax the upper limit on price and only require that $p \geq \underline{p}$. In this case there is always a solution to (5); however, the resulting price might exceed the bid cap $\overline{p}$.

Again, we can implicitly solve for the marginal clearing price as a function of time. However, price is now a function $P : [0, 1] \to [\underline{p}, \infty)$. (In fact, with a linear demand-price relationship, the highest realized price is always below the "choke price" of $N(0)/\gamma$.)

2.7 Profit

By the discussion in 2.6, given a supply function S_i of firm i and also given the supply functions of the other firm, which we will denote by $S_{-i} = (S_j)_{j \neq i}$, we can determine the corresponding price function P. Moreover, at any time t the accrual of profit per unit (normalized) time to firm i is π_{it}:

$$\pi_{it} = S_i(P(t))P(t) - C_i(S_i(P(t))). \tag{7}$$

The profit π_i to firm i over the time horizon is then given by:

$$\begin{aligned} \forall S_j \in \mathbb{S}_j, j = 1, \ldots, n, \pi_i(S_i, S_{-i}) &= \int_{t=0}^{1} \pi_{it} dt, \\ &= \int_{t=0}^{1} S_i(P(t))P(t) - C_i(S_i(P(t)))dt. \end{aligned} \tag{8}$$

That is, the profit π_i is the integral of the profit per unit time over the time horizon.

2.8 Equilibrium definition

Following standard definitions, we make:

Definition 2 A collection of choices $S^\star = (S_i^\star)_{i=1,\ldots,n}$, with $S_i^\star \in \mathbb{S}_i, i = 1, \ldots, n$ is a *Nash supply function equilibrium* (SFE) if:

$$\forall i = 1, \ldots, n, S_i^\star \in \underset{S_i \in \mathbb{S}_i}{\operatorname{argmax}} \{\pi_i(S_i, S_{-i}^\star)\}, \tag{9}$$

where $S_{-i}^\star = (S_j^\star)_{j \neq i}$. □

3 Equilibrium conditions as differential equations

In the following sections we paraphrase and interpret the supply function equilibrium derivations of Klemperer and Meyer [1], Green and Newbery [2], and Green [3], which lead to solutions of the SFE involving the solution of a differential equation. This approach to solving for the SFE as a vector differential equation has been used with considerable success by Green and Newbery in several cases [2, 3]:

1. all firms having the same marginal cost functions and having the same generation capacity constraints, which we refer to as the *symmetric capacitated* case,

2. firms having affine but different marginal cost functions but no capacity constraints, which we refer to as the *asymmetric affine marginal cost uncapacitated* case, and

3. two firms having asymmetric marginal cost functions and capacity constraints, which we refer to as the *asymmetric capacitated duopoly* case.

We develop this approach in order to highlight why an analogous approach is unsuccessful for calculating the SFE in the multi-firm, capacitated, asymmetric case.

3.1 Basic analysis

This section paraphrases the discussion in Klemperer and Meyer [1], Green and Newbery [2], and Green [3] into our notation. The approach in those papers to finding the SFE can be interpreted as:

1. assuming that for each firm i, the supply functions of all the other firms are infinitely differentiable,
2. solving the conditions on price and quantity, at each time t, for maximizing the contribution to profit per unit time for firm i as defined in (7) and
3. finding an infinitely differentiable supply function S_i that matches these conditions, if such a function exists.

We will initially consider a general functional form for the marginal cost function. Consider a firm i and suppose that each other firm $j \neq i$ has committed to an infinitely differentiable supply function S_j. At time t, the price for energy is determined by these supply functions and the production of firm i. Conversely, if firm i is committed to supplying the residual demand at any given price then the price p_t at time t determines the production q_{it} of firm i at time t according to:

$$\forall t \in [0,1], q_{it} = D(t) - \gamma p_t - \sum_{j \neq i} S_j(p_t),$$

where we ignore demand rationing for convenience. Since the supply functions $S_j, j \neq i$ are assumed differentiable, necessary conditions for maximizing the profit per unit time π_{it} at each time t over choices of price p_t are:

$$\forall t \in [0,1], q_{it} = (p_t - C_i'(q_{it}))(\gamma + \sum_{j \neq i} S_j'(p_t)), \tag{10}$$

which we can solve for each t to find a corresponding unique optimal p_t and q_{it} for firm i. If the implicit relationship between q_{it} and p_t is monotonically non-decreasing then we can define a non-decreasing function $S_i : \{p_t | t \in [0,1]\} \to [0, \overline{q}_i]$ that satisfies:

$$\forall t \in [0,1], S_i(p_t) = q_{it}. \tag{11}$$

Applying the implicit function theorem to (10) shows that for each p_t, the function S_i is infinitely differentiable. If, furthermore, each value of q_{it} in (10) satisfies the capacity constraints (3) then we have found a supply function $S_i \in \mathbb{S}_i$ for firm i that achieves the maximum profit per unit time for firm i and each time t, given the supply functions of the other firms. Consequently, this supply function also maximizes the integrated profit π_i for firm i over the time horizon and, moreover, the supply function can be calculated without reference to the load-duration characteristic N.

In summary, we seek a function $S_i \in \mathbb{S}_i$ that satisfies:

$$\forall p \in \mathbb{P}_i, S_i(p) = (p - C_i'(S_i(p)))(\gamma + \sum_{j \neq i} S_j'(p)), \tag{12}$$

where $\mathbb{P}_i = \{p_t | t \in [0,1]\}$; that is, $\mathbb{P}_i$ is the set of all prices for which S_i is defined by (11). An SFE obtains if we can satisfy (12) for every firm i over a common interval of prices. That is, if there are differentiable non-decreasing functions $S_i^\star$ for $i = 1, \ldots, n$, a corresponding price function P, and a set of prices $\mathbb{P} = \{P(t) | t \in [0,1]\}$ satisfying:

$$\forall i = 1, \ldots, n, \forall p \in \mathbb{P}, S_i^\star(p) = (p - C_i'(S_i^\star(p)))(\gamma + \sum_{j \neq i} S_j^{\star\prime}(p)), \tag{13}$$

then $S_i^\star, i = 1, \ldots, n$, is an SFE. This is equation (4) of [3] transcribed into our notation. Somewhat surprisingly, the conditions for the SFE do not depend on the load-duration characteristic N. This has important implications that will be discussed in section 6.

The set $\mathbb{P}$ is an interval because $P(t)$ is a non-decreasing function of t. If $\mathbb{P} = [P(1), P(0)]$ is strictly contained in $[\underline{p}, \overline{p}]$ then we can extend the $S_i^\star$ to being functions on the whole of $[\underline{p}, \overline{p}]$ by defining, for example:

$$\begin{aligned} \forall i = 1, \ldots, n, \forall p \in [\underline{p}, P(1)], S_i^\star(p) &= S_i^\star(P(1)), \\ \forall i = 1, \ldots, n, \forall p \in [P(0), \overline{p}], S_i^\star(p) &= S_i^\star(P(0)). \end{aligned}$$

Klemperer and Meyer [1] characterized the conditions for existence of an SFE in the case of symmetric cost functions with no capacity constraints and discuss the multiplicity of equilibria. In the next section we recall the affine solution in the case of affine marginal costs and no capacity constraints. We then return to the more general asymmetric capacitated case.

3.2 Affine solutions for affine marginal cost functions

In [3, 6, 8], linear and affine SFE are exhibited for the case of affine marginal generation costs of the form (2). The affine SFE $S^{\star\text{affine}} = (S_i^{\star\text{affine}})_{i=1,\ldots,n}$ is of the form:

$$\forall i, \forall p \in \mathbb{P}, S_i^{\star\text{affine}}(p) = \beta_i(p - a_i), \tag{14}$$

where $\beta_i \in \mathbb{R}_+, i = 1, \ldots, n$ satisfies:

$$\forall i, \frac{\beta_i}{1 - c_i \beta_i} = \sum_{j \neq i} \beta_j + \gamma. \tag{15}$$

The affine SFE provides one SFE for the asymmetric affine marginal cost uncapacitated case.

3.3 Manipulation into standard form

If the marginal costs are not affine or if non-affine SFEs are being sought then we must return to the conditions (13). As discussed in [8], these conditions are a set of coupled differential equations that are not in the standard form for differential equations because of the summation of the derivatives in (12). In [8] it was shown that the conditions can be transformed into the following standard form of non-linear vector differential equations:

$$S^{\star\prime}(p) = \left[\frac{1}{n-1}\mathbf{1}\mathbf{1}^\dagger - \mathbf{I}\right] \begin{bmatrix} \frac{S_1^\star(p)}{p - C_1'(S_1^\star(p))} \\ \vdots \\ \frac{S_n^\star(p)}{p - C_n'(S_n^\star(p))} \end{bmatrix} - \frac{\gamma}{n-1}\mathbf{1}, \tag{16}$$

where:

- $S^\star = (S_i^\star)_{i=1,\dots,n}$ is the vector of supply functions and $S^{\star\prime}$ is the derivative of this vector,
- $\mathbf{1}$ is a vector of all ones of length n,
- superscript $\dagger$ means transpose, and
- $\mathbf{I}$ is the identity matrix.

To find an SFE, a natural approach is to seek solutions $S_i^\star$ of the differential equation (16) that also satisfy $S_i^\star \in \mathbb{S}_i$. A natural "initial condition" for the differential equation to implement the price cap condition is (4), which specifies the values of the supply functions at $p = \overline{p}$. The differential equations can then in principle be solved "backwards" from $p = \overline{p}$ to $p = \underline{p}$.

The specification of an initial condition may partly resolve the issue of the multiplicity of equilibria that are typically possible with supply function equilibria. That is, the price cap provides a public signal to the firms that may allow them to coordinate on the equilibrium satisfying $\forall i, S_i^\star(\overline{p}) = \overline{q}_i$, which is presumably the equilibrium that yields the largest profit given the price cap. If the solution of the differential equation for this initial condition is non-decreasing and satisfies the capacity constraints, so that the solution of the differential equation specifies an SFE, and if there is only one such SFE then the SFE may be a plausible outcome for the market.

3.4 Singular equations

A difficulty with solving the differential equation (16) is related to the terms in its right hand side. For each firm i, we define the *marginal cost conditions* to be:

$$\forall p \in [\underline{p}, \overline{p}], C_i'(S_i(p)) \leq p.$$

The marginal cost conditions characterizes prices where a firm i is selling at an operating profit. In numerical experiments, we found that non-affine solutions to the differential equations typically approached the boundary of the marginal cost conditions. That is, the marginal costs approach the price for certain prices. At the boundary of these conditions, the differential equations (16) become singular because of the terms in the denominators of the entries on the right hand side of (16). Nearby to the boundary of the marginal cost conditions, the differential equations become difficult to solve because of numerical conditioning issues.

The singularity can be removed by augmenting the differential equations in a manner analogous to rearranging the equations into parametric form, as discussed for the symmetric, two firm case in [1, §4]. In particular, define a parametric variable u and consider the differential equation:

$$\begin{bmatrix} \dfrac{dS}{du} \\ \dfrac{dp}{du} \end{bmatrix} = \frac{1}{1 + \sum_{i=1}^{n} f_i(S,p)} \begin{bmatrix} f(S,p) \\ 1 \end{bmatrix}, \tag{17}$$

where the function $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ evaluates the right hand side of (16):

$$\forall S \in \mathbb{R}^n, \forall p \in \mathbb{R}, f(S,p) = \left[\frac{1}{n-1}\mathbf{1}\mathbf{1}^\dagger - \mathbf{I}\right] \begin{bmatrix} \frac{S_1}{p - C_1'(S_1)} \\ \vdots \\ \frac{S_n}{p - C_n'(S_n)} \end{bmatrix} - \frac{\gamma}{n-1}\mathbf{1},$$

and where it is understood that if any of the entries of f approach infinity then the ratio on the right hand side of (17) should be evaluated as a limit. The solution of this differential equation yields the relationship of S to p and avoids the singularities of (16). (With a more careful definition of the right hand side of (17), it is also possible to identify u with the normalized time variable.)

3.5 Marginal cost conditions and feasibility constraints

Even with the transformation described in section 3.4 to circumvent the problem of singular equations, the solutions to the differential equations will often reach and even violate the marginal cost conditions. We also found that solutions to the differential equations typically failed to satisfy the feasibility constraints. However, preventing the trajectory from violating the feasibility constraints or the marginal cost conditions poses serious conceptual problems, which we were not able to solve.

We considered a number of approaches to modifying the differential equation to avoid solutions that were not feasible or did not satisfy the marginal cost conditions. For example, we considered imposing the feasibility constraints explicitly in the maximization of profit per unit time to obtain a constrained version of the problem of maximizing profit per unit time. This would modify (10) to include a Lagrange multiplier. The basic difficulty in manipulating the resulting equations into the form of a differential equation is that the dependence of q_{it} on the $S_j', j \neq i$ is no longer invertible. That is, we can longer write an equation analogous to (16) with the derivatives of the supply functions given by a function of the supply functions.

We also tried to model the capacity limit by adding "barrier terms" to the cost function that rapidly increase as the capacity is reached. However, we were not able to reliably generate solutions to the differential equations that satisfied the non-decreasing and capacity constraints.

4 Discussion of assumptions

We discuss some of the assumptions of the model in detail, highlighting three issues that are critical in the the analysis in section 3:

- consistency of bids across the time horizon,
- continuity of the load-duration characteristic, and
- strictly increasing marginal costs.

Firm $i =$	1	2	3
c_i(pounds per MWh per MWh) =	0.5	0.5	0.5
a_i(pounds per MWh) =	9	9	9

Table 1: Cost and capacity data for three firm example system based on [9].

Discussion of these issues will help to clarify where the SFE model is appropriate and where other models, such as Cournot, may be more useful.

In section 3, we already indicated that the marginal cost conditions and the capacity constraints can provide some difficulty in solving the differential equations. To avoid the issues of marginal cost conditions, price caps, and capacity constraints for the discussion in this section, we will concentrate on a symmetric uncapacitated three firm system based on an example in [9]. We first present the example system in section 4.1 and then discuss the issues in sections 4.2–4.4.

4.1 Three firm example system

We consider a three firm electricity market, Based on the example in [9], with each firm having the same cost function. The cost and capacity data is shown in table 1. In the symmetric case, c_i is the same for each firm and a_i is the same for each firm; however, we have kept the notation consistent with (2).

Following [9], we assume a demand slope of $\gamma = 0.125$ GW per (pound per MWh) and a base-case load duration characteristic of:

$$\forall t \in [0, 1], N(t) = 7 + 20(1 - t),$$

with quantities measured in GW. That is, N varies linearly from 27 to 7 GW.

Green and Newbery [2] exhibit the wide range of symmetric equilibria for this symmetric, uncapacitated, no price cap case. The range is defined by the peak demand function:

$$\forall p \in \mathbb{R}_+, D(p, 0) = N(0) - \gamma p.$$

In particular, suppose that the competitive price p_0^{comp} at peak demand is calculated by solving:

$$N(0) - \gamma p_0^{\text{comp}} = \sum_{i=1}^{n} \frac{1}{c_i}(p_0^{\text{comp}} - a_i),$$

and the corresponding quantities are calculated according to:

$$\forall i = 1, \ldots, n, q_i^{\text{comp}} = \frac{1}{c_i}(p_0^{\text{comp}} - a_i).$$

The price p_0^{comp} and the quantities $q_i^{\text{comp}}, i = 1, \ldots, n$ are used as a "competitive initial condition" to solve the differential equations (16) backwards from p_0^{comp} towards $p = \underline{p}$. The solution $S^{\star\text{comp}} = (S_i^{\star\text{comp}})_{i=1,\ldots,n}$ provides one extreme of the range of SFE. We will call $S^{\star\text{comp}}$ the "most competitive symmetric SFE."

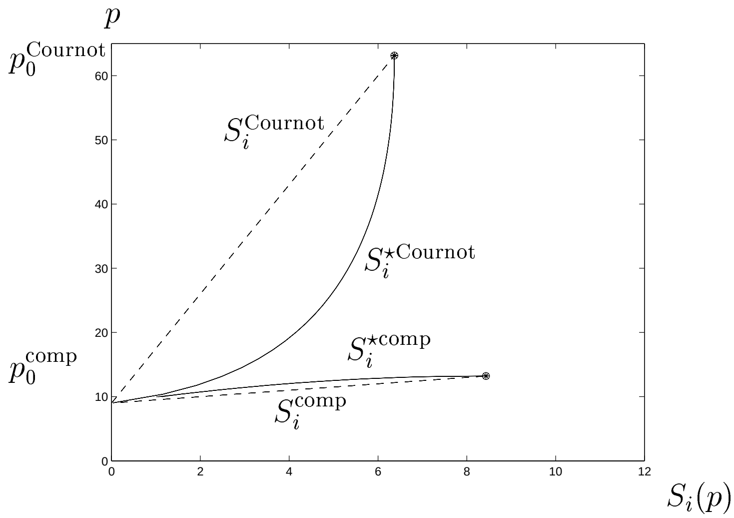

Figure 2: Least and most competitive symmetric SFEs $S_i^{\star\text{Cournot}}$ and $S_i^{\star\text{comp}}$, shown solid, together with Cournot and competitive supply functions S_i^{Cournot} and S_i^{comp}, shown dashed. Source: This figure is based on [2, Figure 3], but uses the data for the symmetric three firm system.

Similarly, Cournot prices p_0^{Cournot} for the peak demand can be calculated by solving:

$$N(0) - \gamma p_0^{\text{Cournot}} = \sum_{i=1}^{n} \frac{1}{(c_i + 1/\gamma)} (p_0^{\text{Cournot}} - a_i).$$

The corresponding quantities are calculated according to:

$$\forall i = 1, \dots, n, q_i^{\text{Cournot}} = \frac{1}{(c_i + 1/\gamma)} (p_0^{\text{Cournot}} - a_i).$$

The price p_0^{Cournot} and the quantities $q_i^{\text{Cournot}}, i = 1, \dots, n$ are used as a "Cournot initial condition" to solve the differential equations (16) backwards from p_0^{Cournot} towards $p = \underline{p}$. The solution $S^{\star\text{Cournot}} = (S_i^{\star\text{Cournot}})_{i=1,\dots,n}$ also satisfies the non-decreasing constraints. The SFE $S^{\star\text{Cournot}}$ provides the other extreme of the range of SFE. We will call $S^{\star\text{Cournot}}$ the "least competitive symmetric SFE."

At each price $p \in [a_i, p_0^{\text{comp}}]$, we have that $S^{\star\text{Cournot}}(p) \leq S^{\star\text{comp}}(p)$ with strict inequality except at $p = a_i$. The most and least competitive symmetric SFEs define a wide range, as illustrated in [2, Figure 3]. Figure 2 is based on [2, Figure 3] and shows the most and least competitive symmetric SFEs for the example system as solid lines. The price p_0^{Cournot} is about six times larger than p_0^{comp} for this example system.

There is a continuum of equilibria intermediate between the most and least competitive symmetric SFEs. These intermediate symmetric SFEs are specified by intermediate choices of starting conditions for the differential equations (16) that are between the competitive and Cournot initial conditions. For example, the affine SFE $S^{\star\text{affine}}$ is intermediate between the most competitive and least competitive symmetric SFEs. For each $p \in [a_i, p_0^{\text{comp}}], S^{\star\text{Cournot}}(p) \leq S^{\star\text{affine}}(p) \leq S^{\star\text{comp}}$, with strict inequality except at $p = a_i$i (unless there is only one firm, in which case $S^{\star\text{Cournot}} = S^{\star\text{affine}}$.)

4.2 Consistency of bids across the time horizon

A fundamental assumption of the analysis in section 3 is that each firm must submit a single non-decreasing supply function that remains valid throughout the time horizon. The coupling effect throughout the time horizon limits the possible equilibria.

In the absence of a requirement to bid consistently over an extended time horizon, there is no such limitation on the range of equilibria. At one extreme, firms could behave as Cournot oligopolists at each time throughout the time horizon. Cournot prices at each time can lead to much higher prices on average than in the supply function equilibrium. At the other extreme, firms could bid competitively at each time throughout the time horizon. If there is no obligation to bid consistently over the time horizon, there is a wide range of possible equilibrium outcomes for each time.

In addition to the equilibrium supply functions, figure 2 also shows two other supply functions:

- "competitive," S^{comp} where the supply functions are the inverses of the marginal cost functions,
- "Cournot," S^{Cournot} where quantities and prices under Cournot competition are calculated for each $t \in [0, 1]$ and a supply function drawn through the resulting price-quantity pairs.

For $a_i < p < p_0^{\text{comp}}$, $S_i^{\star\text{comp}}(p) < S_i^{\text{comp}}(p)$. For $a_i < p < p_0^{\text{Cournot}}$, $S_i^{\text{Cournot}}(p) < S_i^{\star\text{Cournot}}(p)$.

The functions S^{comp} and S^{Cournot} are shown dashed in figure 2. For $n > 1$, S^{Cournot} differs from the SFE $S^{\star\text{Cournot}}$. It is to be emphasized that S^{Cournot} (for $n > 1$) and S^{comp} are not SFEs. (We have ommitted the superscript $\star$ to denote this in the symbols S^{Cournot} and S^{comp}.) The Cournot supply function S^{Cournot} represents an extreme of behavior where each firm behaves as a Cournot oligopolist at each time. The competitive supply function represents the other extreme where each firm behaves competitively at each time. Green and Newbery's analysis shows that when firms must bid a single supply function that applies throughout the time horizon then the range of possible equilibrium outcomes is limited to being between $S^{\star\text{Cournot}}$ and $S^{\star\text{comp}}$. As illustrated in figure 2, this range can be considerably smaller than the range between S^{Cournot} and S^{comp}.

Some analyses implicitly assume that the supply functions apply over time horizons that are much longer than the time between updates of bids allowed under pool rules. For example, [8] models the England and Wales market but the time horizon is considerably longer than a day. This analysis potentially understates the level of market power available to bidders that can update their supply functions arbitrarily day by day or even hour by hour. In the extreme, if firms can update their bids very often then a Cournot model applied at each time may be more appropriate.

Even if there is no explicit requirement to bid consistently, implicit regulatory oversight or the bidders' limited ability to observe the other bidders' supply functions in a timely manner may limit the rapidity with which bids are updated. That is, even if there is no explicit market rule there may be some consistency between bids across time and so supply function equilibrium analysis may be applicable.

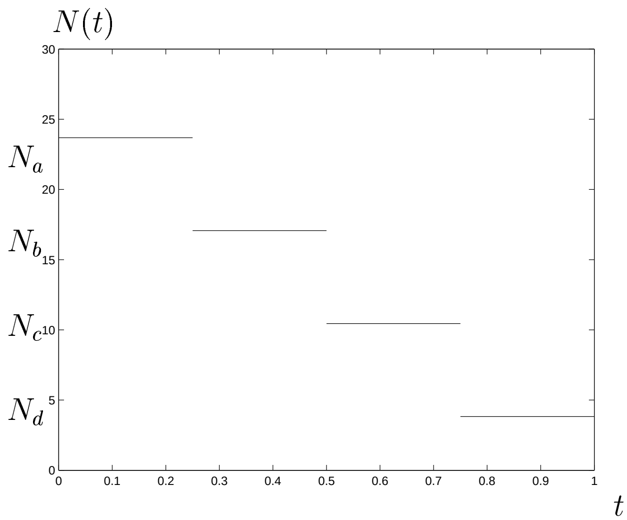

Figure 3: Piece-wise constant load-duration characteristic.

4.3 Continuity of the load-duration characteristic

Even if bid functions are required to be consistent over an extended time horizon, the supply function equilibrium model may not be suitable. For example, suppose that there is no uncertainty in demand over the time horizon. Moreover, suppose that demand is represented by a small number of demand functions, each one applying throughout a period of time in the time horizon. That is, assume that the load-duration characteristic N is piece-wise constant. For example, suppose that there were just, say, four periods, say periods a, b, c, d. Such a piece-wise constant load-duration characteristic is illustrated in figure 3, taking on the values $N_a > N_b > N_c > N_d$.

We can imagine such a load-duration characteristic being used in a day-ahead market with market rules specifying that a clearing price would be calculated for each of the four periods based on the demand function specified for each period. In this case, a supply function consisting of steps could be used to achieve the Cournot outcome in each of the four periods. For example, suppose that the Cournot prices in the four periods were, respectively, $p_a > p_b > p_c > p_d$ and that for firm i the corresponding Cournot quantities were $q_{ia} > q_{ib} > q_{ic} > q_{id}$. Figure 4 shows a bid supply function that will achieve the Cournot prices and quantities. The dashed curve shows the Cournot supply function S_i^{Cournot}. The solid curve shows a bid function that is constant independent of price in each of four price bands around the prices $p_a > p_b > p_c > p_d$. In each band the bid supply is equal to the corresponding Cournot quantities at the prices $p_a > p_b > p_c > p_d$. If each player bids a similar step function then the Cournot outcomes can be achieved in each of the four periods.

In summary, if the demand is specified by a piece-wise constant load-duration characteristic and there is no uncertainty in supply (that is, there are no "forced outages") then we can no longer use the Green and Newbery analysis to argue that the equilibria must be between $S^{\star\text{Cournot}}$ and $S^{\star\text{comp}}$. Given the higher profits available under Cournot behavior, a Cournot model applied in each period separately may be a better predictor of outcomes

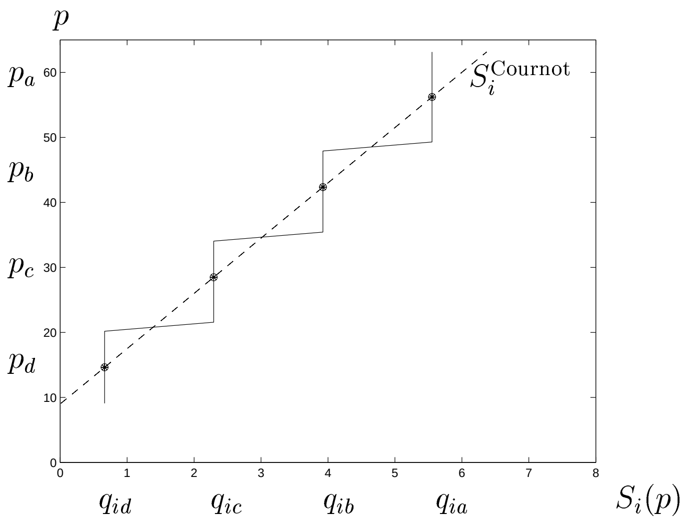

Figure 4: Bid supply function to achieve Cournot prices and quantities in a four period time horizon.

than a supply function equilibrium model that implicitly assumes a continuous variation of demand. We have not proved that the type of supply function shown in figure 4 is an equilibrium in supply functions for a four period market; however, it is a solution that would be easy to maintain with implicit collusion.

In typical day-ahead markets there are usually many more than four demand periods, with 24 or 48 being typical. In this case it may be much more difficult to robustly achieve the Cournot outcome in each period because the bands around each Cournot price will be much smaller. Moreover, uncertainty in each period due to either:

- uncertainty in the demand functions or
- uncertainly in the supply of other firms due to forced outages" of generation,

would prevent the Cournot outcomes from being an equilibrium. For example, in figure 5, there is uncertainty in demand in each of the four demand periods. The uncertainty in each period would prevent the Cournot outcomes from being an equilibrium.

If the demand uncertainty in each period is large enough then the distribution of demand for successive periods can overlap. Similarly, if the uncertainty in the supply of other firms in each period is large enough then the residual demand faced by a firm for successive periods can overlap. With large enough uncertainty in each period, the residual demand faced by a firm would be distributed continuously, even though the market is cleared with a single price applying throughout each period. In this case, the supply function equilibrium is the appropriate equilibrium model.

4.4 Strictly increasing marginal cost functions

In [15], von der Fehr and Harbord argue that "the equilibria found by Green and Newbery (1991) in their model do not generalise to the case in which individual generating sets are of

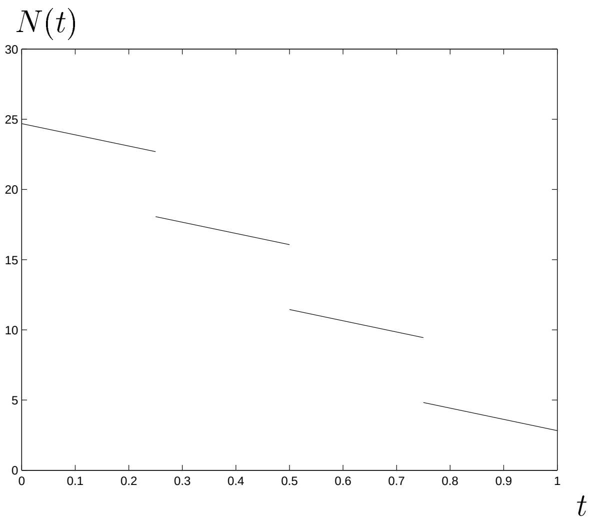

Figure 5: Piece-wise linear load-duration characteristic.

positive size.” That is, if the cost functions reflect economic dispatch of a portfolio of units having finite size, then the Green and Newbery analysis is not applicable. The argument of von der Fehr and Harbord rests on their proposition 1 [15, pp533–534]. However, transcribed into our notation, their proposition assumes that:

$$a_i \neq a_j, 1 \leq i \neq j \leq n, i \tag{18}$$

$$c_i = 0, \forall i, \tag{19}$$

and assumes that all generation by any given generating unit must be offered at a single price. That is, von der Fehr and Harbor's argument rests on the assumptions that each firm has constant marginal cost across its full range of production, that each firm's marginal cost is different from all other firms' marginal costs, and that each individual generating unit must offer all of its capacity at a single price. As discussed in the introduction, the assumption of constant marginal costs is not realistic for a firm that owns a portfolio of generation. While it is true that typical market rules limit the number of “blocks” that can be bid for a given generating unit, there is nevertheless considerable flexibility to offer generation capacity in several blocks having different prices. Moreover, the size of the blocks can usually be modified at will. In summary, von der Fehr and Harbord's criticism of the SFE framework rests on unrealistic assumptions that marginal costs are constant across portfolios, that each portfolio has a different marginal cost, and that fixed block sizes must be bid.

If we assume that (18)–(19) hold then there is no affine SFE solution. To see this, consider an affine function of the form:

$$\forall i, \forall p \in \mathbb{P}, S_i^{\star\text{affine}}(p) = \beta_i p - \alpha_i.$$

We substitute into (12) to obtain,

$$\forall i, \forall p \in \mathbb{P}, \beta_i p - \alpha_i = (p - a_i)(\gamma + \sum_{j \neq i} \beta_j).$$

Since this expression is identically true for all realized prices in the set $\mathbb{P}$, we can equate like coefficients of powers of p. Equating the coefficient of p yields (15) for the particular case $c_i = 0$ for each firm. We obtain:

$$\forall i, \beta_i = \sum_{j \neq i} \beta_j + \gamma.$$

Summing this expression over all firms, yields:

$$\begin{aligned}\sum_{i=1}^{n} \beta_i &= \sum_{i=1}^{n} \sum_{j \neq i} \beta_j + n\gamma, \\ &= (n-1) \sum_{i=1}^{n} \beta_i + n\gamma.\end{aligned}$$

Rearranging, we obtain:

$$(n-2) \sum_{i=1}^{n} \beta_i + n\gamma = 0,$$

which has no solution for $n \geq 2$ in non-negative values of β_i and γ.

This result is not surprising since an affine supply function cannot capture the profit maximizing response, given constant marginal costs, of providing as much production as possible when prices are above marginal cost. However, a slight extension of this argument to the more general continuous but nonlinear SFE case shows that the only SFE that can exist in this situation are solutions that are significantly more competitive than the affine SFE. In summary, when (18)–(19) hold, the range of possible continuous SFEs is severely restricted. (We have not investigated the possible equilibria when discontinuous supply functions can be bid.)

5 Stability of equilibria

In this section, we discuss the stability of equilibria and present conditions for an SFE to be unstable. In practice, an unstable equilibrium is unlikely to be observed. Consequently, we restrict attention to stable equilibria. In [10], Anderson and Xu present conditions for an equilibrium in a similar market structure to be stable. We have not adapted the Anderson and Xu analysis.

To introduce the relevance of stability, recall the symmetric, uncapacitated, no price cap case discussed in section 4.1. As discussed in section 4.1, the range between the symmetric most competitive and symmetric least competitive SFEs can be very wide. We will show, however, that all of the SFEs between the affine SFE $S^{\star\text{affine}}$ and the least competitive symmetric SFE $S^{\star\text{Cournot}}$ are unstable. Consequently, only the SFEs between the most competitive symmetric SFE $S^{\star\text{comp}}$ and the affine SFE $S^{\star\text{affine}}$ will be exhibited in practice. This significantly limits the range of equilibria that can occur in practice.

In section 5.1, we develop the theorem characterizing stability in the context of an SFE where the cost functions are not necessarily symmetric. In section 5.2, we discuss the implications. The theorem as stated applies only to SFEs that are obtained as non-decreasing solutions to the differential equations (16). The reason for this restriction is due to the technical difficulty of characterizing optimal responses when the profit function for a player is non-concave. However, we hypothesize that the theorem holds in much more generality than we have stated it. In particular, the numerical results in sections 9 and 11 are essentially consistent with the conclusion of the theorem.

5.1 Analysis

In this section we first define some particular sets of functions, prove some technical lemmas and then use them in the main theorem. The basic approach involves considering a supply function equilibrium $S^\star = (S_i^\star)_{i=1,\dots,n}$ that is a non-decreasing solution of (16). We then define a perturbation $S_i^\epsilon, i = 1,\dots,n$ of $S_i^\star, i = 1,\dots,n$. In the case that the SFE $S^\star$ is less competitive than the affine SFE, the perturbed functions S_i^ϵ involve "bending" the SFE functions $S_i^\star$ to be slightly more competitive. We then find that the optimal response by firm i to $S_j^\epsilon, j \neq i$ involves an even larger bend. Similarly, in the case that the SFE is more competitive than the affine SFE, the perturbed functions are bent to be slightly less competitive. The optimal response is again an even larger bend. In summary, a small perturbation to the equilibrium results in a response with a larger perturbation so that equilibrium is not stable.

It is relatively easy to construct *an* optimal response by firm i to $S_j^\epsilon, j \neq i$ that deviates more from $S_i^\star$ than does S_i^ϵ. However, there is a continuum of such optimal responses. Most of the technical effort in the the proofs involves showing that *every* optimal response by firm i to $S_j^\epsilon, j \neq i$ deviates more from $S_i^\star$ than does S_i^ϵ.

We begin with:

Definition 3 Suppose that demand is of the form (1). Consider bid supply functions $S_i \in \mathbb{S}_i$ defined on an interval of prices $\mathbb{P} = [\underline{p}, \overline{p}]$. Suppose that supply and demand intersect at the peak demand time $t = 0$ at a price $p_0 \in \mathbb{P}$. We call p_0 the "peak realized price for the bids $S_i, i = 1,\dots,n$." Suppose that supply and demand intersect at the minimum demand time $t = 1$ at a price $p_1 \in \mathbb{P}$. We call p_1 the "minimum realized price for the bids $S_i, i = 1,\dots,n$." □

In the symmetric case, if the players bid the least competitive symmetric equilibrium $S^{\star\text{Cournot}}$ then the peak realized price is p_0^{Cournot}. If the players bid the most competitive symmetric equilibrium $S^{\star\text{comp}}$ then the peak realized price is p_0^{comp}.

Definition 4 Suppose that demand is of the form (1) and that firm i has marginal costs C_i' for $i = 1,\dots,n$. Consider a solution $S_i^\star : \mathbb{P} \to \mathbb{R}, i = 1,\dots,n$ of the differential equation (16) on an interval of prices $\mathbb{P} = [\underline{p}, \overline{p}]$. Suppose that the $S_i^\star$ are non-decreasing and that the peak realized price for the bids $S_i^\star, i = 1,\dots,n$ is $p_0^\star$. By definition of the differential equation, the $S_i^\star$ are continuously differentiable.

Let $\underline{p} < p^\epsilon < p_0^\star$ and define $S^\epsilon : [\underline{p}, \overline{p}] \to \mathbb{R}^n$ by:

$$\forall i = 1,\dots,n, \forall p \in [\underline{p}, \overline{p}], S_i^\epsilon(p) = \begin{cases} S_i^\star(p), & \text{if } \underline{p} \leq p < p^\epsilon, \\ S_i^\star(p^\epsilon) + \beta_i^\epsilon (p - p^\epsilon), & \text{if } p^\epsilon \leq p \leq \overline{p}, \end{cases}$$

where $\beta_i^\epsilon = S_i^{\star\prime}(p^\epsilon), i = 1,\dots,n$. For each firm i, $S_i^\epsilon(p)$ equals $S_i^\star(p)$ for prices p between $\underline{p}$ and p^ϵ. For prices p greater than or equal to p^ϵ, the slope of $S_i^\epsilon(p)$ is constant at $\beta_i^\epsilon = S_i^{\star\prime}(p^\epsilon)$. By definition, S_i^ϵ is continuously differentiable, since $S_i^\star$ is continuously differentiable.

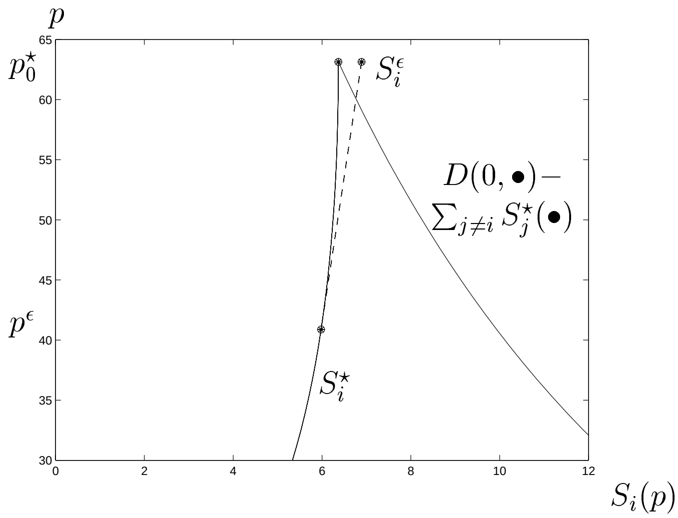

Figure 6: Illustration of definition 4.

We call S_i^ϵ the "linear continuation of $S_i^\star$ from price p^ϵ." We call $S_i^\epsilon(p_0^\star)$ the "maximum relevant supply of the linear continuation of $S_i^\star$." $\Box$

Definition 4 is illustrated in figure 6 for a supply function that is concave. The two solid curves depict the functions:

- $S_i^\star$ and
- the residual demand faced by firm i at peak, $D(0,\bullet) - \sum_{j\neq i} S_j^\star(\bullet)$.

These functions intersect at the point $(p_0^\star, S_i^\star(p_0^\star))$, which is shown as the leftmost of the pair of bullets, $\bullet$, near the top of the figure. The point $(p^\epsilon, S_i^\star(p^\epsilon))$ is illustrated as the bullet that is towards the bottom of the figure. The dashed curve shows the function S_i^ϵ in the interval $[p^\epsilon, p_0^\star]$, with the point $(p_0^\star, S_i^\epsilon(p_0^\star))$ shown as the rightmost of the pair of bullets near the top of the figure.

The supply functions in figure 6 and in all subsequent figures are shown with price p on the vertical axis and the values of production S_i on the horizontal axis. In lemma 4 and subsequently, we will consider supply functions S_i that are strictly concave or strictly convex. Despite the pictorial representation of price versus quantity, when we specify that S_i is concave, for example, we mean that the function S_i is concave as a function of p.

Definition 5 Suppose that demand is of the form (1) and that firm i has marginal costs C_i' for $i = 1, \ldots, n$. Consider a solution $S_i^\star : \mathbb{P} \to \mathbb{R}, i = 1, \ldots, n$ of the differential equation (16) on an interval of prices $\mathbb{P} = [\underline{p}, \overline{p}]$. Suppose that the $S_i^\star$ are non-decreasing and that the peak realized price for the bids $S_i^\star, i = 1, \ldots, n$ is $p_0^\star$.

Let $\underline{p} < p^\epsilon < p_0^\star$ and let S_i^ϵ be the linear continuation of $S_i^\star$ from price p^ϵ. Suppose that firm i faces supply $S_j^\epsilon, j \neq i$. In the following lemma, we will consider one particular profit maximizing feasible and allowable response by firm i to the functions $S_j^\epsilon, j \neq i$. In general there can be a multiplicity of optimal responses by player i. We will construct one such functions and write $\hat{S}_i \in \mathbb{S}_i$ for it. We call $\hat{S}_i(p_0^\star)$ the "maximum relevant supply of the firm i optimal response to $S_j^\epsilon, j \neq i$." □

Lemma 1 *Suppose that demand is of the form (1) and that each firm $i = 1, \ldots, n$ has affine marginal costs C_i' of the form (2) and that the capacity of each firm is arbitrarily large. Consider a solution $S_i^\star : \mathbb{P} \to \mathbb{R}, i = 1, \ldots, n$ of the differential equation (16) on an interval of prices $\mathbb{P} = [\underline{p}, \overline{p}]$. Suppose that the $S_i^\star$ are non-decreasing so that $S_i^\star \in \mathbb{S}_i, i = 1, \ldots, n$ and that the peak realized price for the bids $S_i^\star, i = 1, \ldots, n$ is $p_0^\star$.*

Let $a_i < p^\epsilon < p_0^\star$ and let S_i^ϵ be the linear continuation of $S_i^\star$ from price p^ϵ. We claim that the following function $\hat{S}_i$ is an optimal response to $S_j^\epsilon, j \neq i$:

$$\forall p \in [\underline{p}, \overline{p}], \hat{S}_i(p) = \begin{cases} S_i^\star(p), & \text{if } \underline{p} \leq p < p^\epsilon, \\ S_i^\star(p^\epsilon) + \hat{\beta}_i(p - p^\epsilon), & \text{if } p^\epsilon \leq p \leq \overline{p}, \end{cases} \tag{20}$$

where:

$$\forall i = 1, \ldots, n, \hat{\beta}_i = \frac{\sum_{j \neq i} \beta_j^\epsilon + \gamma}{1 + c_i(\sum_{j \neq i} \beta_j^\epsilon + \gamma)}. \tag{21}$$

Proof As in the derivation of the equilibrium conditions in section 3.1, we first neglect the non-decreasing constraints and consider, for each p, the optimal response of firm i to the bids of the other firms. We then check that the function as defined satisfies the non-decreasing constraints.

We consider the two (just overlapping) intervals of prices $\underline{p} \leq p \leq p^\epsilon$ and $p^\epsilon \leq p \leq p_0^\star$ separately. For prices $\underline{p} \leq p \leq p^\epsilon$, we claim that the quantity $\hat{S}_i(p) = S_i^\star(p)$ is the unique globally optimal response at price p to $\hat{S}_j(p), j \neq p$. This is true by definition of the differential equation (16) because in this range of prices we have that $S_j^{\epsilon\prime} = S_j^{\star\prime}, j \neq i$. This verifies the first line of the right hand side of (20) and, in addition, shows that $\hat{S}_i(p^\epsilon) = S_i^\star(p^\epsilon)$. We will use this last fact to help evaluate terms in the optimal response for prices $p^\epsilon \leq p \leq p_0^\star$. The function $\hat{S}_i$ is continuous at p^ϵ because of the continuity of the derivatives of S_j^ϵ at p^ϵ.

For prices $p^\epsilon \leq p \leq p_0^\star$, the optimality condition (10) states that:

$$\hat{S}_i(p) = (p - a_i - c_i\hat{S}_i(p)) \left(\gamma + \sum_{j \neq i} \beta_j^\epsilon \right).$$

Rearranging this yields the unique globally optimal response at price p of:

$$\hat{S}_i(p) = \hat{\beta}_i(p - a_i),$$

where $\hat{\beta}_i$ is as defined in (21). Substituting in the price $p = p^\epsilon$, we obtain:

$$\hat{S}_i(p^\epsilon) = S_i^\star(p^\epsilon) = \hat{\beta}_i(p^\epsilon - a_i), \tag{22}$$

so that:

$$\begin{aligned} \forall p \in [p^\epsilon, \overline{p}], \hat{S}_i(p) &= \hat{\beta}_i(p - a_i), \\ &= \hat{\beta}_i(p^\epsilon - a_i) + \hat{\beta}_i(p - p^\epsilon), \\ &= S_i^\star(p^\epsilon) + \hat{\beta}_i(p - p^\epsilon), \end{aligned}$$

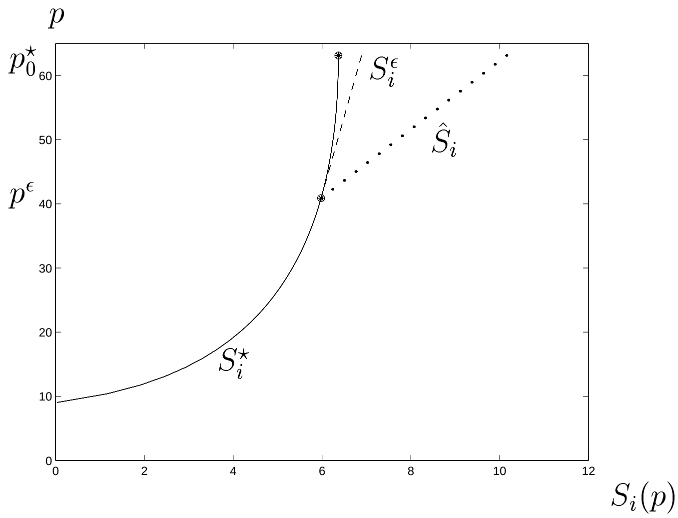

Figure 7: Illustration of lemma 1.

by (22). This verifies the second line of the right hand side of (20).

Now we must check that $\hat{S}_i$, as defined, satisfies the non-decreasing constraints. By definition, $\hat{S}_i$ satisfies the non-decreasing constraints for $\underline{p} \leq p < p^\epsilon$. Moreover, $\hat{S}_i$ satisfies the non-decreasing for $p^\epsilon \leq p \leq \overline{p}$ because $\hat{\beta}_i \geq 0$ since it is the ratio of two positive numbers because $\gamma \geq 0$ and $\beta_i^\epsilon \geq 0$. Since $\hat{S}_i$ is continuous it therefore satisfies the non-decreasing constraints for $\underline{p} \leq p \leq \overline{p}$.

□

Lemma 1 is illustrated in figure 7 for supply functions that are concave. (The price axis is scaled differently to figure 6.) As previously, the function $S_i^\star$ is shown solid. The function S_i^ϵ is shown dashed on the interval $[p^\epsilon, p_0^\star]$ and the function $\hat{S}_i$ is shown dotted on the same interval. The points $(p_0^\star, S_i^\star(p_0^\star))$ and $(p^\epsilon, S_i^\star(p^\epsilon))$ are shown as bullets. Although $S^\star$ is an equilibrium, neither $S^\epsilon = (S_i^\epsilon)_{i=1,\ldots,n}$ nor $(\hat{S}_i)_{i=1,\ldots,n}$ are equilibria. However, for each i, $\hat{S}_i$ is an optimal response to $S_j^\epsilon, j \neq i$.

Definition 6 Suppose that the assumptions of lemma 1 hold. Suppose that firm i bids the function $\hat{S}_i$ while the other firms bid the functions $S_j^\epsilon, j \neq i$. We write $\hat{p}_{0i}$ for the peak realized price for these bids and we write $\hat{p}_{1i}$ for the minimum realized price for these bids. We call $\hat{S}_i(\hat{p}_{0i})$ the "peak realized supply given firm i optimal response to $S_j^\epsilon, j \neq i$." □

Lemma 2 *Suppose that the assumptions of lemma 1 hold. Then the set of all optimal response functions for firm i to $S_j^\epsilon, j \neq i$ is the set of all feasible non-decreasing functions on $[\underline{p}, \overline{p}]$ that match the function $\hat{S}_i$ on the interval $[\hat{p}_{1i}, \hat{p}_{0i}]$, where $\hat{S}_i$ was defined in lemma 1.*

Proof Lemma 1 exhibits one possible optimal response by firm i to the bids $S_j^\epsilon, j \neq i$, namely $\hat{S}_i$. For each price in the interval $[\hat{p}_{1i}, \hat{p}_{0i}]$, the value of $\hat{S}_i(p)$ defined in lemma 1 is the unique globally optimal response at that price. That is, for prices in the interval

$[\hat{p}_{1i}, \hat{p}_{0i}]$, the values of the optimal response for firm i are uniquely determined. However, for prices lower than $\hat{p}_{1i}$ or higher than $\hat{p}_{0i}$, the value of $\hat{S}_i$ is irrelevant because prices outside the interval $[\hat{p}_{1i}, \hat{p}_{0i}]$ are not realized. Any non-decreasing function that matches $\hat{S}_i$ on the interval $[\hat{p}_{1i}, \hat{p}_{0i}]$ will also be an optimal response to $S_j^\epsilon, j \neq i$ so long as the function does not violate the capacity constraints. □

We will be interested in considering an element $\underline{\hat{S}}_i$ of the set of optimal responses to $S_j^\epsilon, j \neq i$ whose maximum value is minimized. This element $\underline{\hat{S}}_i$ will be the closest optimal response to $S_i^\star$ in the sense of a norm to be defined later. One such function $\underline{\hat{S}}_i$ is defined by:

$$\forall p \in [\underline{p}, \overline{p}], \underline{\hat{S}}_i(p) = \min\{\hat{S}_i(p), \hat{S}_i(\hat{p}_{0i})\}, \tag{23}$$

which matches $\hat{S}_i$ on the interval $[\underline{p}, \hat{p}_{0i}]$ but has constant value $\hat{S}_i(\hat{p}_{0i})$ for prices in the interval $[\hat{p}_{0i}, \overline{p}]$.

In the following lemma, we consider the the variation of certain quantities with p^ϵ as it decreases from $p_0^\star$.

Lemma 3 *Suppose that the assumptions of lemma 1 hold. Consider the following expressions:*

- $S_i^\epsilon(p_0^\star)$, *the maximum relevant supply of the linear continuation of* $S_i^\star$,
- $\hat{S}_i(p_0^\star)$, *the maximum relevant supply of the firm i optimal response to* $S_j^\epsilon, j \neq i$.
- $\hat{p}_{0i}$, *the peak realized price given firm i optimal response to* $S_j^\epsilon, j \neq i$, *and*
- $\hat{S}_i(\hat{p}_{0i})$ *the peak realized supply given firm i optimal response to* $S_j^\epsilon, j \neq i$.

In each case, we view the expression as an implicit function of p^ϵ and consider the derivative of it with respect to p^ϵ, evaluated at $p_0^\star$. (Since some of the functions are not defined uniquely for prices greater than $p_0^\star$, strictly speaking we will evaluate the derivative only for movements in the direction of decreasing p^ϵ.) The derivatives of these expressions with respect to p^ϵ evaluated at $p^\epsilon = p_0^\star$ are, respectively, equal to:

- 0,
- $\beta_i^\star - \hat{\beta}_i$,
- $-\dfrac{\beta_i^\star - \hat{\beta}_i}{\sum_{j \neq i} \beta_i^\star + \hat{\beta}_i + \gamma}$, *and*
- $\dfrac{(\beta_i^\star - \hat{\beta}_i)(\sum_{j \neq i} \beta_i^\star + \gamma)}{\sum_{j \neq i} \beta_i^\star + \hat{\beta}_i + \gamma}$,

where:

$$\forall i = 1, \ldots, n, \beta_i^\star = S_i^{\star\prime}(p_0^\star).$$

Proof For the first item, note that:

$$S_i^\epsilon(p_0^\star) = S_i^\star(p^\epsilon) + S_i^{\star\prime}(p^\epsilon)(p_0^\star - p^\epsilon).$$

Totally differentiating with respect to p^ϵ yields:

$$\begin{aligned} \frac{d[S_i^\epsilon(p_0^\star)]}{dp^\epsilon}(p^\epsilon) &= S_i^{\star\prime}(p^\epsilon) + S_i^{\star\prime\prime}(p^\epsilon)(p_0^\star - p^\epsilon) - S_i^{\star\prime}(p^\epsilon), \\ &= S_i^{\star\prime\prime}(p^\epsilon)(p_0^\star - p^\epsilon), \end{aligned}$$

where the double superscript $'$ indicates the second derivative. Evaluating this expression at $p^\epsilon = p_0^\star$ yields zero.

For the second item, note that:

$$\hat{S}_i(p_0^\star) = S_i^\star(p^\epsilon) + \hat{\beta}_i(p_0^\star - p^\epsilon).$$

Differentiating with respect to p^ϵ yields:

$$\frac{d[\hat{S}_i(p_0^\star)]}{dp^\epsilon}(p^\epsilon) = S_i^{\star\prime}(p^\epsilon) + \frac{d\hat{\beta}_i}{dp^\epsilon}(p_0^\epsilon)(p_0^\star - p^\epsilon) - \hat{\beta}_i.$$

Evaluating this expression at $p^\epsilon = p_0^\star$ yields $\beta_i^\star - \hat{\beta}_i$

The third item involves the price that results at peak demand from bids. The price is implicitly determined by the solution of (5). We use the implicit function theorem to show that the price $\hat{p}_{0i}$ is a well-defined function of p^ϵ for p^ϵ in a neighborhood of $p_0^\star$ and to calculate the derivative.

At the peak demand and given that firm i bids $\hat{S}_i$ while the other firms bid the functions $S_j^\epsilon, j \neq i$, equation (5) becomes, after rearranging:

$$\gamma \hat{p}_{0i} + \sum_{j \neq i} S_j^\epsilon(\hat{p}_{0i}) + \hat{S}_i(\hat{p}_{0i}) - N(0) = 0.$$

For $p^\epsilon = p_0^\star$, the solution to this equation is $\hat{p}_{0i} = p_0^\star$. Applying the implicit function theorem we obtain that $\hat{p}_{0i}$ is a well-defined and differentiable function of p^ϵ within a neighborhood of $p_0^\star$. In particular,

$$\frac{d\hat{p}_{0i}}{dp^\epsilon}(p_0^\star) = -\frac{\beta_i^\star - \hat{\beta}_i}{\sum_{j \neq i} \beta_i^\star + \hat{\beta}_i + \gamma}.$$

For the last item, note that:

$$\hat{S}_i(\hat{p}_{0i}) = S_i^\star(p^\epsilon) + \hat{\beta}_i(\hat{p}_{0i} - p^\epsilon),$$

so that

$$\frac{d[\hat{S}_i(\hat{p}_{0i})]}{dp^\epsilon}(p^\epsilon) = S_i^{\star\prime}(p^\epsilon) + \frac{d\hat{\beta}_i}{dp^\epsilon}(p^\epsilon)(\hat{p}_{0i} - p^\epsilon) + \hat{\beta}_i \left(\frac{d\hat{p}_{0i}}{dp^\epsilon}(p^\epsilon) - 1 \right).$$

Evaluating this at $p^\epsilon = p_0^\star$ yields:

$$\begin{aligned} \frac{d[\hat{S}_i(\hat{p}_{0i})]}{dp^\epsilon}(p_0^\star) &= \beta_i^\star + \hat{\beta}_i\left(-\frac{\beta_i^\star - \hat{\beta}_i}{\sum_{j\neq i}\beta_i^\star + \hat{\beta}_i + \gamma} - 1\right), \\ &= \frac{(\beta_i^\star - \hat{\beta}_i)(\sum_{j\neq i}\beta_i^\star + \gamma)}{\sum_{j\neq i}\beta_i^\star + \hat{\beta}_i + \gamma}, \end{aligned}$$

on rearranging. □

Lemma 4 *Suppose that the assumptions of lemma 1 hold. If, for each firm i, $S_i^\star, i = 1, \ldots, n$ is strictly concave on the interval $[a_i, p_0^\star]$ then for each i, $\hat{\beta}_i > \beta_i^\epsilon$. If, for each firm i, $S_i^\star, i = 1, \ldots, n$ is strictly convex on the interval $[a_i, p_0^\star]$ then for each i, $\hat{\beta}_i < \beta_i^\epsilon$.*

Proof We first consider the case where each supply function is strictly concave. Consider the linear function defined for each p by:

$$S_i^\star(p^\epsilon) + \hat{\beta}_i(p - p^\epsilon). \tag{24}$$

This function matches the function $\hat{S}_i$ defined in lemma 1 for prices in the interval $[p^\epsilon, \overline{p}]$. It intersects the function $S_i^\star$ at the point $(p^\epsilon, S_i^\star(p^\epsilon))$. In the proof of lemma 1, it was shown that the function defined in (24) is the same as the function defined for each p by:

$$\hat{\beta}_i(p - a_i).$$

We note that for $p = a_i$, we have that $\hat{\beta}_i(a_i - a_i) = 0$. Also, by definition of (16), $S_i^\star(a_i) = 0$. That is, the function (24) also intersects the function $S_i^\star$ at the point $(a_i, 0)$. In summary, the function (24) has slope $\hat{\beta}_i$ and intersects the increasing, strictly concave function $S_i^\star$ at two points, namely $p = a_i$ and $p = p^\epsilon$, with $a_i < p^\epsilon$. Therefore, $\hat{\beta}_i > S_i^{\star\prime}(p^\epsilon) = \beta_i^\epsilon$.

The argument in the case of each supply function being strictly convex is similar. □

Lemma 4 shows that the relative slopes of the functions S_i^ϵ and $\hat{S}_i$ are as depicted in figure 7 for concave $S_i^\star$.

Corollary 5 *Suppose that the assumptions of lemma 1 hold. First, suppose that for each firm i, $S_i^\star, i = 1, \ldots, n$ is strictly concave on the interval $[a_i, p_0^\star]$. Then the derivatives of the first and fourth quantities considered in lemma 3 are, respectively, zero and negative. As p^ϵ decreases from $p_0^\star$, the fourth quantity becomes strictly greater than the first quantity.*

On the other hand, suppose that for each firm the supply functions are strictly convex on the interval $[a_i, p_0^\star]$. Then the derivatives of the first and fourth quantities considered in lemma 3 are, respectively, zero and positive. As p^ϵ decreases from $p_0^\star$, the fourth quantity becomes strictly less than the first quantity.

Proof Note that for $p^\epsilon = p_0^\star, S_i^\epsilon(p_0^\star) = \hat{S}_i(p_0^\star)$.
□

Finally, we define the notion of unstable equilibrium and characterize conditions for an unstable equilibrium:

Definition 7 Let $\mathbb{S} = \bigtimes_{i=1}^{n} \mathbb{S}_i$ and suppose that $S^\star \in \mathbb{S}$ is an SFE. Let $\|\bullet\|$ be a norm on equivalence classes of elements of $\mathbb{S}$ such that if $S \in \mathbb{S}$ and $\|S - S^\star\| = 0$ then the price function defined by (6) resulting from the supply functions S is the same as the price function resulting from supply functions $S^\star$. Then we say that $S^\star$ is an unstable equilibrium if for every $\epsilon > 0$ there exists $S^\epsilon = (S_i^\epsilon)_{i=1,\ldots,n} \in \mathbb{S}$ such that:

- $\|S^\epsilon - S^\star\| < \epsilon$ and
- if, for each i, $\tilde{S}_i$ is any optimal response to $S_j^\epsilon, j \neq i$ and we define $\tilde{S} = (\tilde{S}_i)_{i=1,\ldots,n}$ then $\left\|\tilde{S} - S^\star\right\| > \|S^\epsilon - S^\star\|$.

□

That is, $S^\star$ is unstable if a small perturbation S^ϵ to $S^\star$ results in responses $\tilde{S}$ by the firms that deviate even more from $S^\star$. "Small perturbation" is defined by a norm on equivalence classes of elements of $\mathbb{S}$ that distinguishes the resulting price functions. The definition is "local in the sense that it does not require that the best response to $\tilde{S}$ be even further from $S^\star$ than $\tilde{S}$.

Theorem 6 *Suppose that the assumptions of lemma 1 hold. Moreover, suppose that either:*

- *for each firm i, $S_i^\star, i = 1, \ldots, n$ is strictly concave on the interval $[a_i, p_0^\star]$ and that the capacity constraints are not binding at the price $p_0^\star$ or*
- *for each firm i, $S_i^\star, i = 1, \ldots, n$ is strictly convex on the interval $[a_i, p_0^\star]$.*

The the SFE $S^\star$ is unstable.

Proof We first consider the case where each supply function Is strictly concave and capacity constraints are not binding. Let $\mathbb{S} = \bigtimes_{i=1}^{n} \mathbb{S}_i$. We define a norm on the equivalence classes of functions in $\mathbb{S}$ that are identical up to the price $p_0^\star$. In particular, define $\|\bullet\|$ by:

$$\forall S \in \mathbb{S}, \|S\| = \max_{i=1,\ldots,n} \int_{\underline{p}}^{p_0^\star} |S_i(p)| \, dp.$$

We show that an arbitrarily small perturbation (in the sense of the norm $\|\bullet\|$) to the SFE $S^\star$ will result in a response by the firms that deviates even more from $S^\star$. This will show that the equilibrium is unstable.

Let $\epsilon > 0$ be given. By continuity of $S^\star$ and $S^{\star\prime}$ in the neighborhood of $p_0^\star$, let $a_i < p^\epsilon < p_0^\star$ be large enough such that:

- $\|S^\epsilon - S^\star\| < \epsilon$, where $S^\epsilon = (S_i^\epsilon)_{i=1,\ldots,n}$,
- By corollary 5, for each $i = 1, \ldots, n$, $\hat{S}_i(\hat{p}_{0i}) > S_i^\epsilon(p_0^\star)$, where:
  - the quantity $S_i^\epsilon(p_0^\star)$ is the maximum relevant supply of the linear continuation of $S_i^\star$ and

– the quantity $\hat{S}_i(\hat{p}_{0i})$ is the peak realized supply given firm i optimal response to $S^\epsilon_j, j \neq i$.

- for each firm $\hat{S}_i(\hat{p}_{0i}) < \overline{q}_i$.

The function S^ϵ represents a perturbation from $S^\star$. By lemma 2, the optimal response of firm i to $S^\epsilon_j, j \neq i$ is any non-decreasing function that matches the function $\hat{S}_i$ on the interval $[\hat{p}_{1i}, \hat{p}_{0i}]$. We show that the functions $\underline{\hat{S}} = (\underline{\hat{S}}_i)_{i=1,\ldots,n}$ defined in (23) are the optimal responses that are closest to $S^\star$ in the sense of the norm $\|\bullet\|$. Moreover, we show that: $\left\|\underline{\hat{S}} - S^\star\right\| > \|S^\epsilon - S^\star\|$.

Because of the concavity of the $S^\star_i$ and by lemma 4, $\hat{S} \geq S^\epsilon \geq S^\star$. Consequently, by the discussion after lemma 2, out of the set of optimal responses by firm i to the bids $S^\epsilon_j, j \neq i$, the function that is closest to $S^\star$ in the sense of the norm $\|\bullet\|$ is the function $\underline{\hat{S}}_i$ defined in (23). We have that:

$$
\begin{aligned}
\forall i = 1,\ldots,n, \forall p \in [p^\epsilon, \hat{p}_{0i}], \underline{\hat{S}}_i(p) &= \hat{S}_i(p), \\
&\geq S^\epsilon_i(p), \text{ by lemma 4}, \\
&\geq S^\star_i(p), \text{ by concavity of } S^\star, \\
\forall i = 1,\ldots,n, \forall p \in [\hat{p}_{0i}, p^\star_0], \underline{\hat{S}}_i(p) &= \underline{\hat{S}}_i(\hat{p}_{0i}), \text{ by (23)}, \\
&= \hat{S}_i(\hat{p}_{0i}), \text{ by (23)}, \\
&> S^\epsilon_i(p^\star_0), \text{ by construction}, \\
&\geq S^\epsilon_i(p), \text{ since } S^\epsilon_i \text{ is non-decreasing}, \\
&\geq S^\star_i(p).
\end{aligned}
$$

Also:

$$\forall i = 1,\ldots,n, \forall p \in [\underline{p}, p^\epsilon], S^\epsilon_i(p) = S^\star_i(p).$$

Consequently, by definition of the norm, $\left\|\underline{\hat{S}} - S^\star\right\| > \|S^\epsilon - S^\star\|$. Moreover, every vector $\tilde{S}$ of optimal responses to S^ϵ satisfies: $\left\|\tilde{S} - S^\star\right\| \geq \left\|\underline{\hat{S}} - S^\star\right\| > \|S^\epsilon - S^\star\|$ and so the equilibrium $S^\star$ is unstable.

In the case of strictly convex S_i, $\hat{p}_{0i} > p^\star_0$ and the optimal response to S^ϵ is $\hat{S}$ throughout the interval $[\underline{p}, p^\star_0]$. □

Lemma 7 *Consider a solution $S^\star$ of (16) for demand of the form (1) and affine marginal costs of the form (2). Then:*

$$S^{\star\prime\prime}(p) = \left[\frac{1}{n-1}\mathbf{1}\mathbf{1}^\dagger - \mathbf{I}\right] \begin{bmatrix} \frac{S^{\star\prime}_1(p)(p-a_i)-S^\star_1(p)}{(p-C'_1(S^\star_1(p)))^2} \\ \vdots \\ \frac{S^{\star\prime}_n(p)(p-a_n)-S^\star_n(p)}{(p-C'_n(S^\star_n(p)))^2} \end{bmatrix}. \tag{25}$$

Proof Differentiate (16). □

Corollary 8 *Consider any symmetric non-decreasing solution $S^\star$ of (16) for demand of the form (1) and affine marginal costs of the form (2) where the marginal costs are the same for each firm. Suppose that either:*

- *the solution satisfies* $S^\star < S^{\star\text{affine}}$ *and the capacity constraints are strictly satisfied at all prices up to the peak realized price* $p_0^\star$ *or*
- *the solution satisfies* $S^\star > S^{\star\text{affine}}$.

Then $S^\star$ *is unstable.*

Proof If the solution satisfies $S^\star < S^{\star\text{affine}}$ then the terms in the denominator of the right hand side of (25) are all negative so that $\forall i = 1, \ldots, n, \forall p \in [a_i, p_0^\star], S_i^{\star\prime\prime}(p) < 0$ and so the supply functions are strictly concave. Furthermore, by assumption the capacity constraints are strictly satisfied.

If the solution satisfies $S^\star > S^{\star\text{affine}}$ then the terms in the denominator of the right hand side of (25) are all positive so that $\forall i = 1, \ldots, n, \forall p \in [a_i, p_0^\star], S_i^{\star\prime\prime}(p) > 0$ and so the supply functions are strictly convex. In either case, the SFE is not stable. □

5.2 Discussion

Corollary 8 shows that in the symmetric case every SFE between $S^{\star\text{Cournot}}$ and $S^{\star\text{affine}}$ (including $S^{\star\text{Cournot}}$ but not including $S^{\star\text{affine}}$) is unstable unless capacity constraints are just binding at the peak realized price. The corollary also shows that in the symmetric case every SFE between $S^{\star\text{affine}}$ and $S^{\star\text{comp}}$ (including $S^{\star\text{comp}}$ but not including $S^{\star\text{affine}}$) is unstable. Baldick and Kahn show that, under mild conditions, if the bid functions are required to be affine, then the affine SFE $S^{\star\text{affine}}$ is stable in the function space of affine SFEs [8]. We hypothesize the stronger result that, with respect to a suitable norm on $\mathbb{S}$, the affine SFE is stable in $\mathbb{S}$. Although there is a wide range of equilibria in the symmetric unconstrained case, this wide range is unlikely to be observed in practice because the equilibria that are different to $S^{\star\text{affine}}$ are unstable.

The situation is illustrated in figure 8 for the three firm example system discussed in section 4.1. Green and Newbery's analysis [2] suggests that any equilibrium between the least competitive symmetric SFE $S^{\star\text{Cournot}}$ and the most competitive symmetric SFE $S^{\star\text{comp}}$ can be observed. These supply functions are both shown solid in figure 8. However, corollary 8 shows that only the affine SFE $S^{\star\text{affine}}$ (shown dashed in figure 8) can be stable. Only stable equilibria are likely to be observed in practice.

Green and Newbery [2] use the least competitive SFE $S^{\star\text{Cournot}}$ for some of their analysis to estimate an upper bound on price mark-ups in the England and Wales system. Their calculations yield price mark-ups that are much higher than were observed. Corollary 8 suggests that $S^{\star\text{Cournot}}$ is not a tight bound on the equilibrium mark-ups.

We can also consider applying the previous analysis to SFEs that are not obtained as solutions to the differential equation and where the profit function is non-concave. In this case we can only guarantee that the response $\hat{S}_i$ that we construct to the bids $S_j^\epsilon, j \neq i$ is a local but not necessarily globally optimal response. Nevertheless, even in the case that the profit function is non-concave, if the functions $S_i^\star$ are all concave or all convex then a similar construction can still be used to find a function $\hat{S}_i$ that is a better response than S_i^ϵ to the bids $S_j^\epsilon, j \neq i$. However, we cannot in general show that $\hat{S}_i$ is the *best* response to the bids $S_j^\epsilon, j \neq i$. This suggests, however, that if:

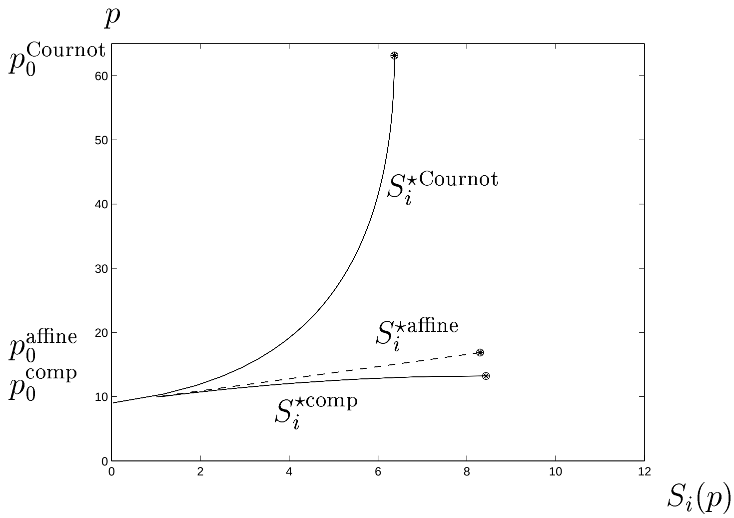

Figure 8: Illustration of corollary 8.

- the non-decreasing constraints are binding on an SFE,
- supply functions are all strictly concave or all strictly convex in the vicinity of the maximum realized price, and
- the capacity constraints are not binding (in the case of strictly convex supply functions),

then the equilibrium will be unstable. Moreover, if a local improvement algorithm is used by firms to respond to the supplies of other firms then such equilibria will not be observed in practice. Conversely, we expect that, in the vicinity of the peak price, stable equilibria will not involve all strictly concave supply functions unless capacity constraints are binding and will not involve all strictly convex supply functions.

The construction of bending the supply function also fails if the capacity constraints are binding. Supply functions satisfying the differential equation (16) that are less competitive than $S^{\star\text{affine}}$ can be stable only if the capacity constraints are just binding at the peak realized price. For example, Day and Bunn [9] use their numerical technique on the symmetric three firm case upon which our example is based and exhibit results that are consistent with the least competitive equilibrium $S^{\star\text{Cournot}}$. They apparently choose the capacity constraints to be exactly equal to the Cournot supply at the peak demand. That is, they are implicitly considering a capacitated case where the capacity constraints are just binding at the peak realized price. The least competitive symmetric equilibrium $S^{\star\text{Cournot}}$ is stable in that constrained case.

Finally, we observe that the stability results may not apply perfectly to equilibria calculated using the numerical framework that we will develop in section 8. This is because the proofs of equilibria being unstable rely on the construction of arbitrary differentiable functions. In the numerical framework we develop, we will use a finite dimensional parametrization of the supply functions. We have not investigated theoretically the conditions for an unstable equilibrium in this context, but speculate that the results would be less "clear cut" than the results we have developed here.

6 Allowable functions

Klemperer and Meyer [1] and Green and Newbery [2] show that if the cost functions are the same for each firm and if a non-affine symmetric solution is obtained for the differential equation (16) then for sufficiently high prices the solution will either violate the non-decreasing constraints (in the case of solutions that are less competitive than the affine SFE) or become vertical. However, so long as the realized prices do not exceed the price at which the solutions become decreasing or vertical then the solution of the differential equation provides an SFE.

In this section we will observe that it is generally very difficult to find solutions of (16) that are non-decreasing over all realized prices except in very special cases, namely:

- if the cost functions are the same for each firm, as explored by Klemperer and Meyer [1] and Green and Newbery [2],
- if the marginal costs are affine and there are no capacity constraints so that there are linear or affine solutions to (16), which was explored in [3, 6, 8], or
- if the load factor over the time horizon is very close to 100%.

In the general case, of firms having capacity constraints and asymmetric costs, solutions of (16) typically violate the non-decreasing requirements somewhere over the range of realized prices over the time horizon. The following theorem helps to explain why this is the case. It shows that the solutions of the differential equation must satisfy tight bounds in order for the solution to be non-decreasing over a range of prices. The theorem partially generalizes analysis in Klemperer and Meyer developed for the symmetric case [1, Proposition 1].

6.1 Analysis

Theorem 9 *Consider a solution $S_i^\star : \mathbb{P} \to \mathbb{R}, i = 1, \ldots, n$ of the differential equation (16) on an interval of prices $\mathbb{P} = [\underline{p}, \overline{p}]$. If each function $S_i^\star, i = 1, \ldots, n$ is non-decreasing on $\mathbb{P}$ then:*

$$\forall i = 1, \ldots, n, \forall p \in \mathbb{P}, \gamma \leq \frac{S_i^\star(p)}{p - C_i'(S_i^\star(p))} \leq \left(\frac{1}{n-1}\right) \sum_{j=1}^{n} \left\{ \frac{S_j^\star(p)}{p - C_j'(S_j^\star(p))} \right\} - \frac{\gamma}{n-1}. \quad (26)$$

Proof We first prove the lower bound condition in (26). That is, we prove:

$$\forall i = 1, \ldots, n, \forall p \in \mathbb{P}, \gamma \leq \frac{S_i^\star(p)}{p - C_i'(S_i^\star(p))}.$$

The differential equation (16) collects together and rearranges the conditions (12) applied to each firm. Rearranging (12), we obtain:

$$\begin{aligned}
\frac{S_i^\star(p)}{p - C_i'(S_i^\star(p))} &= \gamma + \sum_{j \neq i} S_j'(p), \\
&\geq \gamma,
\end{aligned}$$

since $S_j'(p) \geq 0, \forall j$ by assumption.

We now prove the upper bound condition in (26). That is, we prove:

$$\forall i = 1, \ldots, n, \forall p \in \mathbb{P}, \frac{S_i^\star(p)}{p - C_i'(S_i^\star(p))} \leq \left(\frac{1}{n-1}\right) \sum_{j=1}^{n} \left\{ \frac{S_j^\star(p)}{p - C_j'(S_j^\star(p))} \right\} - \frac{\gamma}{n-1}.$$

Let $\mathbf{1}_i$ be the vector of all zeros, except in the i-th place where it is equal to 1. For any $p \in \mathbb{P}$,

$$\begin{aligned}
0 &\leq S_i^{\star\prime}(p), \\
&= [\mathbf{1}_i]^\dagger S^{\star\prime}(p), \\
&= \frac{1}{n-1}\mathbf{1}^\dagger \begin{bmatrix} \frac{S_1^\star(p)}{p - C_1'(S_1^\star(p))} \\ \vdots \\ \frac{S_n^\star(p)}{p - C_n'(S_n^\star(p))} \end{bmatrix} - \frac{S_i^\star(p)}{p - C_i'(S_i^\star(p))} - \frac{\gamma}{n-1}, \text{ by (16)}, \\
&= \left(\frac{1}{n-1}\right) \sum_{j=1}^{n} \frac{S_j^\star(p)}{p - C_j'(S_j^\star(p))} - \frac{S_i^\star(p)}{p - C_i'(S_i^\star(p))} - \frac{\gamma}{n-1}.
\end{aligned}$$

Rearranging we obtain:

$$\frac{S_i^\star(p)}{p - C_i'(S_i^\star(p))} \leq \left(\frac{1}{n-1}\right) \sum_{j=1}^{n} \left\{ \frac{S_j^\star(p)}{p - C_j'(S_j^\star(p))} \right\} - \frac{\gamma}{n-1}.$$

□

6.2 Discussion

In theorem 9, the lower bound condition in (26) requires that γ be no larger than the smallest entry of the vector:

$$\begin{bmatrix} \frac{S_1^\star(p)}{p - C_1'(S_1^\star(p))} \\ \vdots \\ \frac{S_n^\star(p)}{p - C_n'(S_n^\star(p))} \end{bmatrix}. \tag{27}$$

Furthermore, the expression:

$$\left(\frac{1}{n-1}\right) \sum_{j=1}^{n} \left\{ \frac{S_j^\star(p)}{p - C_j'(S_j^\star(p))} \right\},$$

is equal to $\frac{n}{n-1}$ times the average of the entries in the vector (27). The upper bound condition in (26) in theorem 9 requires that each entry of the vector (27) is smaller than $\left(\frac{n}{n-1}\right)$ times the average of the entries. For n large, the ratio $\left(\frac{n}{n-1}\right)$ is only slightly greater than one. That is, the upper bound condition in theorem 9 dictates that the values of $\frac{S_j^\star(p)}{p-C_j'(S_j^\star(p))}$ must fall in a narrow range in order for the solution to the differential equation be non-decreasing.

In the cases of:

1. symmetric cost functions and symmetric solutions to the differential equations or
2. affine solutions to the differential equations with affine marginal costs,

then the necessary conditions in theorem 9 are relatively mild as we will discuss in the following two sections. We will then discuss capacity constraints.

6.2.1 Symmetric cost functions

If the cost functions and the solutions to the differential equation are symmetric then the upper bound condition in (26) can be verified as follows:

$$\begin{aligned}
&\frac{S_i^\star(p)}{p-C_i'(S_i^\star(p))} \\
&= \left(\frac{n-1}{n-1}\right)\frac{S_i^\star(p)}{p-C_i'(S_i^\star(p))} \\
&= \left(\frac{1}{n-1}\right)\sum_j \frac{S_j^\star(p)}{p-C_j'(S_j^\star(p))} - \frac{1}{n-1}\frac{S_i^\star(p)}{p-C_i'(S_i^\star(p))}, \\
&\qquad \text{since the cost functions and solutions are symmetric,} \\
&\leq \left(\frac{1}{n-1}\right)\sum_j \left\{\frac{S_j^\star(p)}{p-C_j'(S_j^\star(p))}\right\} - \left(\frac{\gamma}{n-1}\right),
\end{aligned}$$

where the inequality is true if the lower bound condition in (26) in theorem 9 is satisfied. That is, the upper bound condition on $\frac{S_i^\star(p)}{p-C_i'(S_i^\star(p))}$ is automatically satisfied if the lower bound condition is satisfied. This means that the non-decreasing constraints are easier to satisfy in the symmetric case than in the asymmetric case. In fact, as Klemperer and Meyer show [1, Proposition 1], a necessary and sufficient condition for a symmetric solution of the differential equations to be an SFE is that the lower bound condition in (26) be satisfied. In the symmetric case, the equilibrium supply functions $S^{\star\text{Cournot}}$ and $S^{\star\text{comp}}$ satisfy the non-decreasing constraints over the range of realized prices. Moreover every symmetric equilibrium between these equilibria also satisfies the non-decreasing constraints.

6.2.2 Affine solutions for affine marginal cost functions

The affine SFE $S^{\star\text{affine}}$ was exhibited in (14). Each function $S_i^{\star\text{affine}}$ has slope $\beta_i \in \mathbb{R}_+$ satisfying (15). Since the $\beta_i \in \mathbb{R}_+$, the affine functions are guaranteed to be non-decreasing.

6.2.3 Capacity constraints

To interpret theorem 9 in the case of capacity constraints (3), we will assume that the marginal costs effectively increase very rapidly as capacity constraints are approached. This

Firm $i =$	1	2	3	4	5
c_i(pounds per MWh per MWh) =	2.687	4.615	1.789	1.93	4.615
a_i(pounds per MWh) =	12	12	8	8	12

Table 2: Cost data based on five firm industry described in [8].

means that entries in the vector (27) change rapidly with p as capacity constraints are approached so that the upper bound condition will not be satisfied unless all firms reach their capacity at the same price. We conjecture that this in unlikely except in the case of symmetric cost functions and capacities. That is, in the asymmetric capacitated case, the solution to the differential equation will typically violate the non-decreasing constraints at some price.

6.3 Five firm example system

To illustrate theorem 9, we consider a five firm example system based on the cost data presented in [8] for the five strategic firm industry in England and Wales subsequent to the 1999 divestiture. Table 2 shows the cost parameters. Firms 2 and 5 have identical cost functions.

Using the analysis in section 6.2.2, we find that the slopes of the affine solutions are:

$$\beta = \begin{bmatrix} 0.2840 \\ 0.1857 \\ 0.3718 \\ 0.3550 \\ 0.1857 \end{bmatrix},$$

and that the affine SFE is given by:

$$\forall p \in [12, \infty), S^{\star\text{affine}}(p) = \begin{bmatrix} 0.2840(p-12), \\ 0.1857(p-12), \\ 0.3718(p-8), \\ 0.3550(p-8), \\ 0.1857(p-12) \end{bmatrix}. \tag{28}$$

(For prices below $p = 12$ pounds per MWh, the minimum capacity constraint is binding on firms 1, 2, and 5, so we only define the affine solution for $p \geq 12$ pounds per MWh. A piece-wise affine SFE for this case is derived in [8] and described in detail in section 11.2.)

Using any initial condition for (16) of the form $(\overline{p}, S^{\star\text{affine}}(\overline{p}))$, with $\overline{p} > 12$ pounds per MWh, will yield an affine solution that is identical to $S^{\star\text{affine}}$. For example, using $\overline{p} =$ 30 pounds per MWh and integrating backwards yields figure 9. (In this figure and most subsequent figures illustrating the five firm example, firm 1 is shown as a dashed line, firms 2 and 5 are shown superimposed as a dash-dot line, firm 3 is shown as a solid line and firm 4 is shown as a dotted line.) The numerical solution of the differential equation differs very

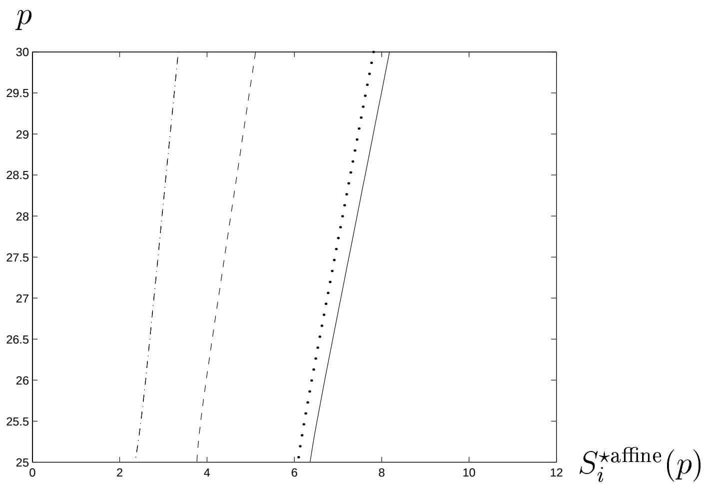

Figure 9: Solution of (16) that matches affine SFE. Firm 1 is shown as a dashed line, firms 2 and 5 are shown superimposed as a dash-dot line, firm 3 is shown as a solid line and firm 4 is shown as a dotted line.

slightly from (28) because of numerical conditioning issues in the solution of the differential equations. However, the correspondence with the exact affine solution is very close.

To illustrate that the solution of (16) will violate the non-decreasing constraints when the solution is non-affine, we considered initial conditions that differed only very slightly from the initial condition of $\overline{p} = 30$ pounds per MWh and $S^{\star\text{affine}}(\overline{p})$. In particular, we considered the 32 vertices of the hypercube whose vertices are specified by:

$$S_i(\overline{p}) = 0.999 \times S_i^{\star\text{affine}}(\overline{p}), 1.001 \times S_i^{\star\text{affine}}(\overline{p}), i = 1, \ldots, 5.$$

That is, we successively decreased and increased each entry in $S_i^{\star\text{affine}}(\overline{p})$ by 0.1% and used the resulting vector as the initial condition to integrate backwards from $p = \overline{p}$.

The results of integrating from these 32 initial conditions are shown in figure 10. Each initial condition was integrated from $\overline{p}$ backwards until a price p' was reached where the non-decreasing constraints were violated significantly for one of the firms. In each case, the trajectory for all five firms was plotted for $[p', \overline{p}]$. Since the values of p' varied with the initial condition, the trajectories for most of the initial conditions can be individually distinguished in figure 10.

As previously, firms 2 and 5 have identical costs. Their trajectories are shown as the leftmost bundle of curves in figure 10. Whenever firms 2 and 5 are started with different initial conditions, the resulting trajectories for them will diverge. Firms 3 and 4 are the rightmost pair of bundles of curves in figure 10. Firm 1 appears as the middle bundle of curves in figure 10.

As shown in this figure, for every one of the 32 initial conditions, the supply of either firm 1 or firm 2 or firm 5 violates the non-decreasing constraints for some prices between 26.5 and 30. In summary, in this example the differential equation (16) yields solutions that violate the non-decreasing constraints when the initial conditions differ even slightly from satisfying the affine SFE conditions. Although this is not a proof in general, it suggests why solutions of (16) may violate the non-decreasing constraints.

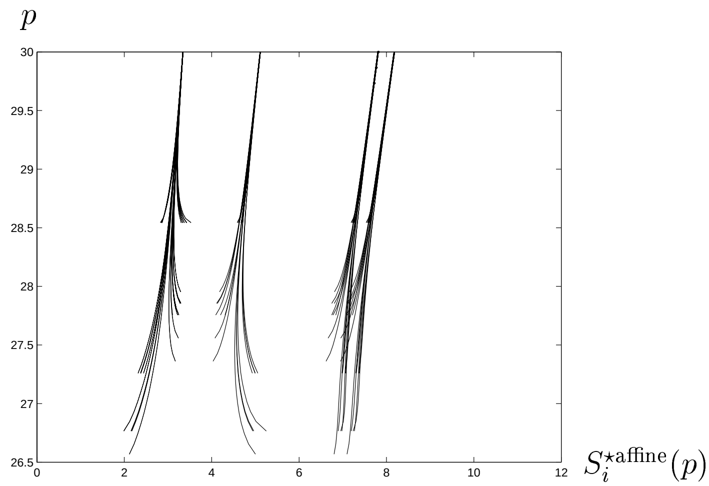

Figure 10: Solution of (16) from 32 initial conditions that are slight perturbations of a point satisfying the affine SFE.

6.4 Summary

The most serious difficulty with the differential equation approach to solving for the SFE is that the differential equations do not "automatically" satisfy the capacity or non-decreasing conditions. Theorem 9 implies that unless the cost functions are all very similar or there are no capacity constraints then the non-decreasing constraints will typically be violated in a solution of the differential equations, unless the range of realized prices is small enough to only cover a segment of the solution that happens to be non-decreasing. The example in section 6.3 shows that even a very slight deviation from the affine solution results in solutions of (16) that are non-decreasing only over a narrow range of prices. If the load factor over the time horizon were very close to 100% then such a solution of (16) would be an equilibrium. However, if the load factor is significantly below 100% then most such solutions would violate the non-decresing constraints over the range of realized prices.

This analysis provides two observations. First, the usual approach to solving differential equations to obtain the SFE may not work in the case of heterogeneous portfolios of generation with capacity constraints when the load factor deviates significantly from 100%. In this case, we must explicitly impose the non-decreasing constraints.

Second, as discussed in the introduction, a basic criticism of the SFE approach is that there are multiple equilibria. Certainly, if *every* possible specification of the initial conditions for the differential equations (16) yielded an equilibrium then this extreme multiplicity of equilibria would limit the predictive value of the SFE approach. However, when the load factor deviates significantly from 100%, many of these putative equilibria are ruled out by the non-decreasing constraints. This strengthens the observations by Klemperer and Meyer in [1] that were made for the symmetric case concerning the multiplicity of equilibria. Moreover, the price cap condition (4), when it is binding on the behavior of firms, further limits the range of potential equilibria.

Solutions such as shown in figure 10 could form part of an equilibrium only if either:

1. the range of realized prices was very restricted, or,

2. there were a discontinuity in the derivative of the supply functions.

The first case could occur if the load factor were close to 100%. In this case, there would be a multiplicity of equilibria, with the range depending on the range of the function N, but not on the detailed dependence of $N(t)$ on t. Conversely, extended time horizons having load factors well below 100% rule out many of the solutions of (16) from being supply functions.

In the second case, we can imagine a discontinuous change in the behavior of the firms due to, for example, a binding capacity constraint being reached at a particular price. In this case, we can imagine equilibrium solutions consisting of the union of solutions of (16) that are "pasted" together at various break-points. We will confirm this observation theoretically in the next section and then see in section 11 that the numerical solutions have this appearance.

7 Strict satisfaction of non-decreasing constraints

In this section we show that although it is necessary to represent the non-decreasing constraints, they will be strictly satisfied at typical equilibria. The intuition behind this apparently paradoxical observation is that once the non-decreasing constraints are enforced, the profit maximizing response of a firm is strictly increasing. If the non-decreasing constraints were relaxed then the profit maximizing response would no longer be increasing because of the non-concavity in the profit function. This observation allows us to characterize SFEs in more detail. In section 7.2 we illustrate these observations with a two firm example.

7.1 Analysis

We first make some definitions to clarify the nature of "binding constraints."

Definition 8 Consider supply functions S and suppose that $\mathbb{P}$ is the interval of realized prices corresponding to S. Also suppose that for some firm i and for some interval $[\hat{p}, \check{p}] \subset \mathbb{P}$ we have that:

1. $S_j, j \neq i$ is differentiable on $[\hat{p}, \check{p}]$,
2. S_i is constant on $[\hat{p}, \check{p}]$, with $0 < S_i(p) = q_i < \overline{q}_i, \forall p \in [\hat{p}, \check{p}]$, and
3. the profit function is increasing with price in the interior of the interval in the following sense:
$$\forall p \in (\hat{p}, \check{p}), q_i - (p - C'(q_i)(\gamma + \sum_{j \neq i} S'_j(p)) > 0.$$

Then we say that the non-decreasing constraints are *manifestly binding* for firm i on $[\hat{p}, \check{p}]$. □

Definition 9 Consider supply functions S and suppose that $\mathbb{P}$ is the interval of realized prices corresponding to S. Suppose that for some firm i and for some interval $[\hat{p}, \check{p}] \subset \mathbb{P}$ we have that:
$$\forall p \in (\hat{p}, \check{p}), S_i(p) - (p - C'(S_i(p))(\gamma + \sum_{j \neq i} S'_j(p)) = 0.$$

Then we say that the non-decreasing constraints are *not apparently binding* for firm i on $[\hat{p}, \check{p}]$. □

Definition 10 Consider supply functions S and suppose that for firm i, S_i is the optimal non-decreasing response to $S_j, j \neq i$. Consider relaxing the non-decreasing constraints on the supply function of firm i. If the globally optimal response of firm i to $S_j, j \neq i$, given the relaxed constraints, is not equal to S_i then we say that non-decreasing constraints are *actually binding* for firm i. □

The adjective "manifestly" is used in definition 8 to emphasize that the choice of the supply function has been palpably restricted by the non-decreasing constraints. Definition 9 of "not apparently binding" covers the case where the the choices of supply function for firm i locally maximize the profit function for a given price. Definition 10 of "actually binding" covers the case where relaxing the non-decreasing constraints would cause a different response. In principle, this could occur because either:

- the non-decreasing constraints were manifestly binding or
- the non-decreasing constraints were not apparently binding but yet the non-decreasing constraints ruled out other responses having higher profits.

The following theorem shows that under relatively mild conditions the non-decreasing constraints cannot be manifestly binding. We show that it is impossible for the non-decreasing constraints to be:

- not apparently binding up to some price $\hat{p}$ and
- manifestly binding for prices above $\hat{p}$.

That is, it is impossible for the supply function to become "flat" over an interval of prices. Moreover, this means that the non-decreasing constraints will always be not apparently binding.

As we will show in the example in section 7.2, the non-decreasing constraints can be actually binding. The conclusion is that while the non-decreasing constraints will be not apparently binding they will, however, be actually binding.

Theorem 10 *Let $\gamma > 0$. Consider continuous and piece-wise continuously differentiable supply functions S and suppose that $\mathbb{P}$ is the interval of realized prices corresponding to S. Also suppose that for firm i, S_i is the optimal non-decreasing response to $S_j, j \neq i$. Consider prices $\tilde{p}, \hat{p}, \check{p} \in \mathbb{P}$ such that either:*

- *$\tilde{p} < \hat{p} < \check{p}$ or*
- *$\hat{p}$ is equal to the minimum realized price and $\tilde{p} = \hat{p} < \check{p}$.*

Suppose that the non-decreasing constraints are not apparently binding for firm i on $[\tilde{p}, \hat{p}]$. (If $\tilde{p} = \hat{p}$ this condition is null.) Then the non-decreasing constraints cannot be manifestly binding for firm i on $[\hat{p}, \check{p}]$.

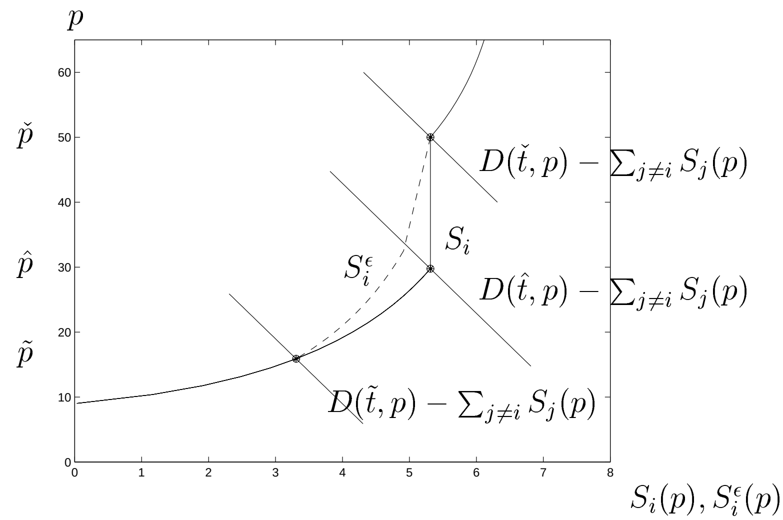

Figure 11: The functions S_i (shown solid) and S_i^ϵ (shown dashed) defined in proof of theorem 10.

Proof Suppose that the non-decreasing constraints were manifestly binding for firm i on $[\hat{p}, \check{p}]$. By adjusting $\tilde{p}$ and $\check{p}$ if necessary we can assume that $S_j, j = 1, \ldots, n$ are continuously differentiable on the intervals $[\tilde{p}, \hat{p})$ and $(\hat{p}, \check{p}]$. (That is, the functions $S_j, j = 1, \ldots, n$ may fail to be continuously differentiable only at $p = \hat{p}$.) The situation is shown in figure 11. The function S_i is illustrated with the solid line. (Note that the function S_i is drawn on the horizontal axis while its argument is drawn on the vertical axis.)

Let $P : [0, 1] \to \mathbb{P}$ be the realized prices at each time in the time horizon. Let $\tilde{t} > \hat{t} > \check{t}$ be the times corresponding to $\tilde{p}, \hat{p}, \check{p}$, respectively. The residual demand $D(t, p) - \sum_{j\neq i} S_j(p)$ faced by firm i at times $t = tildet, \hat{t}, \check{t}$ is also shown.

Consider a parameter $\epsilon \geq 0$ and the following construction of functions $S_i^\epsilon : [\underline{p}, \overline{p}] \to [0, \overline{p}_i]$ and $P^\epsilon : [0, 1] \to [\underline{p}, \overline{p}]$. The functions S_i^ϵ and P^ϵ are parametrized by ϵ.

First, for each $p \in [\underline{p}, \tilde{p}]$ and each $p \in [\check{p}, \overline{p}]$, let $S_i^\epsilon(p) = S_i(p)$, so that S_i^ϵ matches S_i except on the interval $[\tilde{p}, \check{p}]$. Similarly, for each $t \in [0, \check{t}]$ and each $t \in [\tilde{t}, 1]$ let $P^\epsilon(t) = P(t)$.

Second, for each $t \in [\check{t}, \hat{t}]$ find p such that:

$$N(t) - \gamma p - \sum_{j\neq i} S_j(p) = q_i - \epsilon \left(\frac{t - \check{t}}{\hat{t} - \check{t}} \right), \tag{29}$$

and define $S_i^\epsilon(p) = q_i - \epsilon \left(\frac{t-\check{t}}{\hat{t}-\check{t}} \right)$ and $P^\epsilon(t) = p$. (By assumption, since $\gamma > 0$, the left hand side of (29) is strictly decreasing with p so that there is a solution.) By construction, note that $p \in [P^\epsilon(\hat{t}\,), \check{p}] \subset [\hat{p}, \check{p}]$ and that S_i^ϵ is non-decreasing on $[P^\epsilon(\hat{t}\,), \check{p}]$ and that S_i^ϵ is continuous at $p = \check{p}$. The function S^ϵ is shown dashed in figure 11. Also P^ϵ is non-increasing on $[0, \hat{t}]$.

Furthermore, by the implicit function theorem we have that the derivatives of these functions with respect to ϵ, evaluated at $\epsilon = 0$ are, respectively:

$$\begin{aligned}
\forall t \in [\check{t}, \hat{t}], \frac{d[P^\epsilon(t)]}{d\epsilon}(0) &= \frac{1}{\gamma + \sum_{j \neq i} S'_j(P(t))} \left(\frac{t - \check{t}}{\hat{t} - \check{t}} \right), \\
&\geq 0, \\
\forall t \in [\check{t}, \hat{t}], \frac{d[S_i^\epsilon(P^\epsilon(t))]}{d\epsilon}(0) &= -\left(\frac{t - \check{t}}{\hat{t} - \check{t}} \right), \\
&= -\left(\gamma + \sum_{j \neq i} S'_j(P(t) \right) \frac{d[P^\epsilon(t)]}{d\epsilon}(0).
\end{aligned}$$

Third, for each $t \in [\hat{t}, \tilde{t}]$ find p such that:

$$N(t) - \gamma p - \sum_{j \neq i} S_j(p) = S_i - \epsilon \left(\frac{\tilde{t} - t}{\tilde{t} - \hat{t}} \right),$$

and define $S_i^\epsilon(p) = S_i - \epsilon \left(\frac{\tilde{t} - t}{\tilde{t} - \hat{t}} \right)$ and $P^\epsilon(t) = p$. By construction, note that $p \in [\tilde{p}, P^\epsilon(\hat{t}\,)] \subset [\tilde{p}, \check{p}]$ and that S_i^ϵ is non-decreasing on $[\tilde{p}, P^\epsilon(\hat{t}\,)]$ and that S_i^ϵ is continuous at $p = \tilde{p}$ and at $p = P^\epsilon(\hat{t}\,)$. Also, The function S^ϵ is shown dashed in figure 11. P^ϵ is non-increasing on $[\hat{t}, 1]$.

Again, by the implicit function theorem we have that the derivatives of these functions with respect to ϵ, evaluated at $\epsilon = 0$ are, respectively:

$$\begin{aligned}
\forall t \in [\check{t}, \hat{t}], \frac{d[P^\epsilon(t)]}{d\epsilon}(0) &= \frac{1}{\gamma + \sum_j S'_j(P(t))} \left(\frac{\tilde{t} - t}{\tilde{t} - \hat{t}} \right), \\
&\geq 0, \\
\forall t \in [\check{t}, \hat{t}], \frac{d[S_i^\epsilon(P^\epsilon(t))]}{d\epsilon}(0) &= -\left(\gamma + \sum_{j \neq i} S'_j(P(t) \right) \frac{d[P^\epsilon(t)]}{d\epsilon}(0).
\end{aligned}$$

We now consider the change in profit accruing to firm i by changing its bid from S_i to S_i^ϵ. In particular, we calculate the derivative of the profit with respect to ϵ, evaluated at

$\epsilon = 0$. We have that:

$$
\begin{aligned}
&\frac{d[\pi_i]}{d\epsilon}(0) \\
&= \frac{d\left[\int_{t=\check{t}}^{\tilde{t}} \pi_{it} dt\right]}{d\epsilon}(0), \\
&\quad \frac{d\left[\int_{t=\check{t}}^{\tilde{t}} S_i^{\epsilon}(P^{\epsilon}(t))P^{\epsilon}(t) - C_i(S_i^{\epsilon}(P^{\epsilon}(t)))dt\right]}{d\epsilon}(0), \\
&= \int_{t=\check{t}}^{\tilde{t}} \left[S_i(P(t))\frac{d[P^{\epsilon}(t)]}{d\epsilon}(0) + (P(t) - C_i'(S_i(P(t)))\frac{d[S_i^{\epsilon}(P^{\epsilon}(t))]}{d\epsilon}(0)\right] dt, \\
&= \int_{t=\check{t}}^{\hat{t}} \frac{1}{\gamma + \sum_{j\neq i} S_j'(P(t))} \left[q_i - (P(t) - C_i'(q_i)) \left(\gamma + \sum_{j\neq i} S_j'(P(t))\right)\right] \left(\frac{t-\check{t}}{\hat{t}-\check{t}}\right) dt \\
&\quad + \int_{t=\hat{t}}^{\tilde{t}} \frac{1}{\gamma + \sum_j S_j'(P(t))} \\
&\quad \times \left[S_i(P(t)) - (P(t) - C_i'(S_i(P(t)))) \left(\gamma + \sum_{j\neq i} S_j'(P(t))\right)\right] \left(\frac{t-\check{t}}{\hat{t}-\check{t}}\right) dt, \\
&= \int_{t=\check{t}}^{\hat{t}} \frac{1}{\gamma + \sum_{j\neq i} S_j'(P(t))} \left[q_i - (P(t) - C_i'(q_i)) \left(\gamma + \sum_{j\neq i} S_j'(P(t))\right)\right] \left(\frac{t-\check{t}}{\hat{t}-\check{t}}\right) dt \\
&\quad + \int_{t=\hat{t}}^{\tilde{t}} \frac{1}{\gamma + \sum_j S_j'(P(t))}[0] \left(\frac{t-\check{t}}{\hat{t}-\check{t}}\right) dt, \\
&\quad \text{since the non-decreasing constaints are not apparently binding for firm } i \text{ on } [\tilde{p}, \hat{p}], \\
&= \int_{t=\check{t}}^{\hat{t}} \frac{1}{\gamma + \sum_{j\neq i} S_j'(P(t))} \left[q_i - (P(t) - C_i'(q_i)) \left(\gamma + \sum_{j\neq i} S_j'(P(t))\right)\right] \left(\frac{t-\check{t}}{\hat{t}-\check{t}}\right) dt, \\
&> 0,
\end{aligned}
$$

since the integrand is strictly positive over almost all of the interval $[\check{t}, \hat{t}]$ because the non-decreasing constraints are manifestly binding on $[\hat{p}, \check{p}]$. But this contradicts the hypothesis that S_i is an optimal response to $(S_j)_{j\neq i}$. Contradiction. □

Corollary 11 *Suppose that $S^\star$ is an SFE with each function $S_i^\star, i = 1, \ldots, n$ piece-wise continuously differentiable on the range of realized prices $\mathbb{P}$. Consider a firm i and prices $\tilde{p}, \hat{p}, \check{p} \in \mathbb{P}$ such that either:*

- *$\tilde{p} < \hat{p} < \check{p}$ or*
- *$\hat{p}$ is equal to the minimum realized price and $\tilde{p} = \hat{p} < \check{p}$.*

Suppose that the non-decreasing constraints are not apparently binding for firm i on $[\tilde{p}, \hat{p}]$. (If $\tilde{p} = \hat{p}$ this condition is null.) Then the non-decreasing constraints cannot be manifestly binding for firm i on $[\hat{p}, \check{p}]$. □

The following corollary allows us to characterize SFEs:

Corollary 12 *Consider a piece-wise continuously differentiable SFE* $S^\star = (S_i^\star)_{i=1,\dots,n}$*. Consider any interval* $[\hat{p}, \check{p}]$ *of prices such that:*

- *the* $S^\star$ *are continuously differentiable,*
- *the capacity constraints of firms* $i_1, i_2, \dots, i_m$ *are not binding, and*
- *the capacity constraints of the other firms are binding.*

Then the supplies of firms $i_1, i_2, \dots, i_m$ *on* $[\hat{p}, \check{p}]$ *match a solution of (16) where instead of having* n *firms with cost functions* $C_1, \dots, C_n$*, respectively, there are* m *firms with cost functions given by the cost functions* $C_{i_1}, C_{i_2}, \dots, C_{i_m}$ *of the* m *firms* $i_1, i_2, \dots, i_m$*.*

Proof Note that by corollary 11, the non-decreasing constraints cannot be manifestly binding for firms $i_1, i_2, \dots, i_m$ on $[\hat{p}, \check{p}]$. Since the supply functions are continuously differentiable on this interval, they must satisfy the optimality conditions (10). But rearranging these optimality conditions, and noting that $S_j^{\star\prime}(p) = 0$ for $p \in [\hat{p}, \check{p}]$ and $j \neq i_1, i_2, \dots, i_m$, we find that $S_i^\star, i = i_1, i_2, \dots, i_m$ must satisfy an m firm version of (16). □

Corollary 12 allows us to characterize piece-wise continuously differentiable SFEs. In particular, as suggested in section 6.4, such SFEs involve the pasting together of solutions of (16). The points of non-differentiability in the SFE occur where the solutions of (16) for adjacent intervals are pasted together. Unfortunately, since we do not in general know where the break-points of the pieces of $S^\star$ will lie, we cannot usually use corollary 12 to directly construct an SFE. Because the solutions in each interval satisfy (16), it is only the range of the load-duration characteristic N, and not its exact functional form, that determines the possible equilibria as shown in:

Corollary 13 *The set of possible piece-wise continuously differentiable equilibria depends on the range of the load-duration characteristic but not on its exact form.*

Proof Consider a piece-wise continuously differentiable SFE $S^\star$ corresponding to a load-duration characteristic N_1. By assumption, we can partition the range of realized prices into intervals such that $S^\star$ is continuously differentiable on the interior of the interval and is a non-decreasing solution of (16).

Suppose that N_2 is another load-duration characteristic that has the same range as N_1. But since the range of N_2 is the same as the range of N_1, $S^\star$ is piece-wise continuously differentiable and non-decreasing over the (identical) range of realized prices for N_2. That is, $S^\star$ is an SFE corresponding to the load-duration characteristic N_2. □

Although the set of equilibria is independent of the exact functional form of the load-duration characteristic, in a numerical framework where we consider convergence to (one particular) equilibrium, it may be the case that the form of the load-duration characteristic affects which of the equilibria is exhibited by the numerical framework.

7.2 Two firm example system

To see the implications of theorem 10 and its corollaries, we will consider the following two firm market. To motivate the necessity of explicitly representing the non-decreasing constraints, we will postulate a supply function for firm 2 and then consider the optimal reaction of firm 1.

The demand is:

$$\forall p \in \mathbb{R}_+, \forall t \in [0,1], D(p,t) = 20 + 4.6(1-t) - 0.1p.$$

Firm 2 has a maximum capacity of $\overline{q}_2 = 17.1$ and is assumed to have bid a supply function of:

$$\forall p \in \mathbb{R}_+, S_2(p) = \begin{cases} 0.9p, & \text{if } p \leq 19, \\ 17.1, & \text{if } p > 19. \end{cases}$$

This function is non-decreasing. In the context of a multi-firm market, we can also think of S_2 as being the aggregate supply of all firms besides firm 1.

The cost function for firm 1 is:

$$\forall q_1 \in \mathbb{R}_+, C_1(q_1) = \frac{1}{7}(q_1)^2 + 4q_1,$$

with marginal cost $C_1'(q_1) = \frac{2}{7}(q_1) + 4$. We will assume that firm 1 has the same capacity as firm 2, so that $\overline{q}_1 = 17.1$. We will consider the optimal response of firm 1 to the given supply function of firm 2. (The resulting pair of supply functions S_1 and S_2 is not necessarily an equilibrium unless we make further assumptions but serves to illustrate the importance of the non-decreasing constraints.)

7.2.1 Ignoring the non-decreasing constraints

We first consider the optimal response by firm 1, ignoring the non-decreasing constraints. This simply amounts to maximizing the profit per unit time for firm 1 at each time. To maximize the profit per unit time to firm 1 for various times, we first observe that the profit function is *piece-wise* concave, with the pieces defined by whether or not the price is above $p = 19$. In fact, for some times t, the profit per unit time has two local maxima and so we must search over both pieces to find the value of q_{1t} that globally maximizes the profit per unit time of firm 1. We will consider the conditions for maximizing profit per unit time at two particular times: namely $t = 0$ and $t = 1$. This will suffice to demonstrate that a function S_1 that globally maximizes profit at each price would not be non-decreasing.

For $t = 1$, the maximum profit per unit time for firm 1 in the region $p \leq 19$ occurs for $p_1 = 13$ and $q_{11} = 7$. The corresponding profit is $\pi_{11} = 56$. For the region $p > 19$, it can be verified that the profit is always decreasing with p, and the profit is continuous across the regions as a function of p. Therefore, the globally optimal profit occurs at $p_1 = 13$ and $q_{11} = 7$.

At $t = 0$, the maximum profit per unit time for firm 1 in the region $p \leq 19$ occurs for $p \approx 15.59$, with corresponding profit of 92.83. For prices $p > 19$, the profit is maximized for $p_0 = 40$, with corresponding quantity $q_{10} = 3.5$ and profit 124.25. Therefore, the globally optimal profit $\pi_{10} = 124.25$ occurs at $p_0 = 40$ and $q_{10} = 3.5$.

The significance of this example is that if we seek to use the pairs (p_0, q_{10}) and (p_1, q_{11}) to define points in the supply function S_1 for firm 1, we have just found that the resulting function will violate the non-decreasing constraint. The example relies on the particular choice of cost and demand, but many similar choices will yield similar results. For example, Anderson and Philpott [18] provide another example.

A further complicating issue is that we must perform a maximization over a non-concave profit function, so that the necessary conditions obtained from differentiating the profit are not sufficient. In general, at any price where the supply function of a firm $j \neq i$ changes slope, there will be a break-point (and potentially a non-concavity) in the profit function for firm i.

In this example, the break-point in the profit function of firm 1 is due to the change in the slope of the supply function of firm 2 as it reaches its full capacity $\overline{q}_2$. Such a break-point can also occur due to the capacity constraints of fringe firms. This issue prompted an *ad hoc* approach in [8].

7.2.2 Including non-decreasing constraints

We now consider the optimal response S_1 of firm 1 to S_2 considering the non-decreasing constraints. Assume a price cap of $\overline{p} = 50$. To approximate the optimal response of firm 1, we approximate the function space $\mathbb{S}_1$ by a subspace $\underline{\mathbb{S}}_1$ of $\mathbb{S}_1$ and choose S_1 from $\underline{\mathbb{S}}_1$. We specify $\underline{\mathbb{S}}_i$ as the set of piece-wise linear non-decreasing continuous functions with break-points at $p = 0, 4, 10, 13, 16, 19, 40, 50$. Since the marginal cost of firm 1 at zero production is 4, the optimal response of firm 1 must involve zero production up to price $p = \underline{p} = 4$. At a price of $p = \overline{p} = 50$, we specify that the firm must produce at full output $\overline{q}_1$, so this leaves the values $S_1(10), S_1(13), S_1(16), S_1(19)$ and $S_1(40)$ of the supply function at prices $p = 10, 13, 16, 19, 40$ to be specified. For the resulting supply function to be non-decreasing, we impose:

$$0 \leq S_1(10) \leq S_1(13) \leq S_1(16) \leq S_1(19) \leq S_1(40) \leq \overline{q}_1.$$

We calculated the profit π_1 of firm 1 according to (8), given the assumptions on demand and S_2. Exact integration was used. The profit is not concave as a function of $S_1(10), S_1(13), S_1(16), S_1(19)$, and $S_1(40)$. For example, figure 12 shows profits versus choices of $S_1(19)$ and $S_1(40)$ for $S_1(10) = S_1(13) = S_1(16) = 1$. The maximum profit point given $S_1(10) = S_1(13) = S_1(16) = 1$ is shown as a bullet. Maximum profit occurs for $S_1(19) = 1, S_1(40) = 5$. The profit curves up as $S_1(19)$ decreases.

Because of the non-concavity of the profit function, we used a grid search to find the (approximate) globally optimal choice for $S_1(10), S_1(13), S_1(16), S_1(19), S_1(40)$. We found that the maximum profit occurs for $S_1(13) \approx 7, S_1(16) \approx 9$, with the realized prices being contained in the interval $[13, 16]$. Moreover, given $S_1(13) = 7, S_1(16) = 9$, the profit π_1 is independent of $S_1(10), S_1(19)$, and $S_1(40)$ for values of $S_1(10), S_1(19)$, and $S_1(40)$ that satisfy:

$$0 \leq S_1(10) \leq S_1(13), S_1(16) \leq S_1(19) \leq S_1(40) \leq \overline{q}_1.$$

Figure 13 shows the profit π_1 of firm 1 for $S_1(10) = 1, S_1(19) = S_1(40) = 17$ and versus choices of $S_1(13)$ and $S_1(16)$ in the range $1 \leq S_1(13) \leq S_1(16) \leq 17$. The maximum profit point is shown as a bullet.

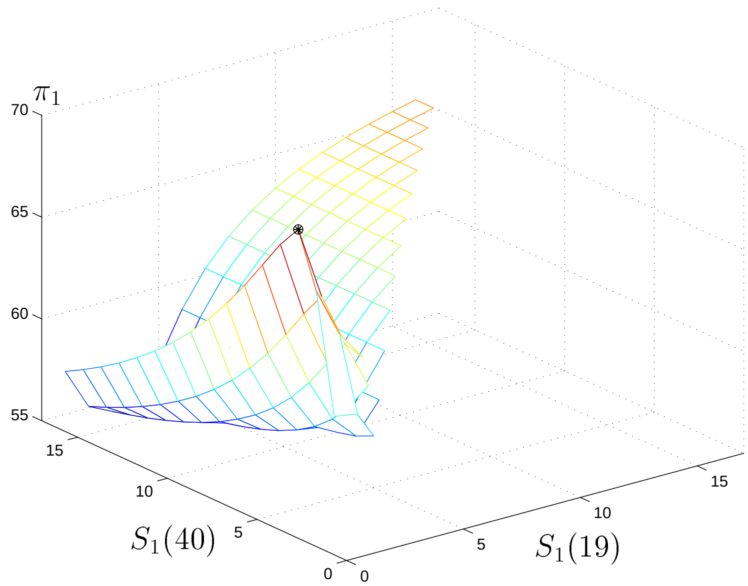

Figure 12: Profit for firm 1 for $S_1(10) = S_1(13) = S_1(16) = 1$. and versus choices of $S_1(19)$ and $S_1(40)$ in the range $1 \leq S_1(19) \leq S_1(40) \leq 17$.

Since the optimal response satisfies $S_1(13) \approx 7 < S_1(16) \approx 9$, the optimal supply function of firm 1 is strictly increasing. That is, the non-decreasing constraints are not manifestly binding over the range of realized prices. That is, S_1 satisfies theorem 11. However, the discussion in section 7.2.1 shows that the optimal response would change if the non-decreasing constraints were relaxed for firm 1. That is, the non-decreasing constraints are actually binding.

As demonstrated by figure 12, the profit function for firm 1 is not concave as a function of $S_1(10), S_1(13), S_1(16), S_1(19), S_1(40)$ when the supply function is piece-wise linear with break-points at $p = 4, 10, 13, 16, 19, 40$. The profits for small values of $S_1(19)$ and $S_1(40)$ bend up as $S_1(19)$ approaches zero. *A fortiori* the profit of firm 1 is not concave as a function of $S_1 \in \mathbb{S}_1$. However, the integration of the profit function over time in (8) has "smeared" out the non-concavities of the underlying profit per unit time functions. In particular, recalling the optimal behavior for firm 1 *just considering time* $t = 0$, we found previously that firm 1 should bid a quantity $q_{10} = 3.5$ at a price of $p_0 = 40$. That is, $S_1(40) = 3.5$, which would require that $S_1(19) \leq 3.5$ to satisfy the non-decreasing constraint. This strategy corresponds to values of $(S_1(19), S_1(40))$ that are near the origin in figure 12. However, the implications of this choice at other times is to significantly reduce the overall profit: for this reason, larger values of $S_1(19)$ and $S_1(40)$ actually yield the global optimum profit for firm 1.

In the next section we discuss an approach to numerically estimating equilibria when the cost functions are asymmetric, while taking explicit account of the non-decreasing and capacity constraints and the price cap. This will allow us to empirically investigate the issue of multiplicity of equilibria. We will see that the implications of the theorem proved in section 7.1 are corroborated by the numerical results:

- the solutions are piece-wise differentiable and appear to match solutions of (16) between points of non-differentiability;
- the non-decreasing constraints are never manifestly binding over the range of realized

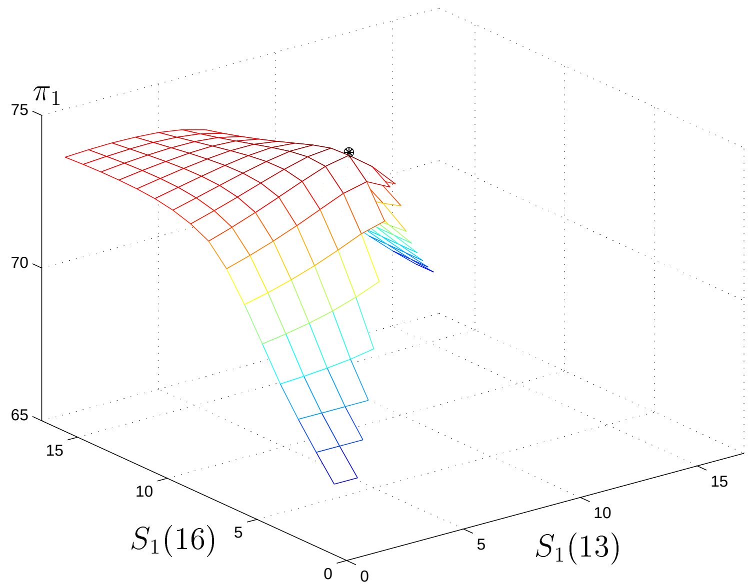

Figure 13: Profit for firm 1 versus choices of $S_1(13)$ and $S_1(16)$ in the range $1 \leq S_1(13) \leq S_1(16) \leq 17$.

prices; however, the non-decreasing constraints are actually binding and their representation is essential in order to calculate the equilibria.

8 Iterations in function space

Because of the difficulties with the differential equations approach to seeking the SFE in general, we take an iterative numerical approach. Such numerical approaches can usually be expected to yield only stable equilibria, unless started at an equilibrium or unless the iterative process produces a particular iterate that happens to be an equilibrium. In the following sections, we describe the step direction, updated, and step size and the computational issues involved.

8.1 Step direction

Given a current estimate of the equilibrium supply functions, denoted $S_i^{(\nu)}$ at iteration ν, we calculate the following step directions:

$$\forall i, \Delta S_i^{(\nu)} \in \underset{\Delta S_i}{\operatorname{argmax}} \{\tilde{\pi}_i(S_i^{(\nu)} + \Delta S_i, S_{-i}^{(\nu)}) | S_i^{(\nu)} + \Delta S_i \in \underline{\mathbb{S}}_i\}, \tag{30}$$

where:

- $\tilde{\pi}_i$ is an approximation to π_i,
- $S_{-i}^{(\nu)} = (S_j^{(\nu)})_{j \neq i}$, and
- $\underline{\mathbb{S}}_i$ is a finite dimensional convex subset of $\mathbb{S}_i$.

8.2 Supply function subspace

The set $\mathbb{S}_i$ consisted of piece-wise linear non-decreasing functions with break-points evenly spaced between $(\underline{p} + 0.1)$ pounds per MWh and $(\overline{p} - 0.1)$ pounds per MWh, where $\underline{p}$ is the price minimum and $\overline{p}$ is the price cap. At $p = \underline{p}$, we define $S_i(p) = 0$. At $p = \overline{p}$, we require $S_i(p) = \overline{q}_i$. That is, $\mathbb{S}_i$ is convex.

For most cases, we used 40 break-points. We also tested some of the cases using functions with other numbers of break-points to investigate whether any of the results were an artifact of the number of break-points.

8.3 Update and step size

An initial guess $S_i^{(0)}, i = 1, \ldots, n$ was used as a starting function to begin the iterations. We then update the iterates according to:

$$\forall \nu, \forall i, S_i^{(\nu+1)} = S_i^{(\nu)} + \alpha \Delta S_i^{(\nu)},$$

where $\alpha \in (0, 1]$ is a step-size. Since $S_i^{(\nu)}$ and $S_i^{(\nu)} + \Delta S_i^{(\nu)}$ are both elements of the convex set $\mathbb{S}_i$, then so is $S_i^{(\nu+1)}$.

We tested several step-size rules, including an elaborate "Armijo"-like rule [19] that sought to find directions at each iteration that guaranteed improvement in the profit of all firms. However, we found that a fixed step-size of $\alpha = 0.1$ performed satisfactorily.

Day and Bunn [9] take a similar approach, except that they only find an approximate local maximizer of (30) at each iteration and use a step size of $\alpha = 1$ at each iteration. Their approach requires less effort per iteration, but because of the inflexibility of the unity step size does not appear to converge [9, §4].

8.4 Profit function approximation

We estimated the integral in the profit function by dividing the time horizon into intervals having end-points at:

- $t = 0$,
- the times corresponding to the realized prices at the break-points of the supply function, and
- $t = 1$.

Linear interpolation was used to find the prices corresponding to $t = 0$ and $t = 1$, while (5) was used to evaluate the time corresponding to each price break-point. (If a price break-point corresponded to a "negative" time or to a time greater than one, it was simply discarded. Only realized prices, that is, prices for which $0 \leq t \leq 1$ in (5), are relevant in calculating the profit over the time horizon in (8).)

In some cases, we used the trapezoidal rule to approximate the integral on each interval. In other cases, we integrated the quadratic function on each interval exactly.

8.5 Computational issues

Iterating in the function space of supply functions requires considerable computational effort at each iteration and is subject to the drawback that the problem of finding the search direction may have multiple local optima. In practice, we use an iterative local search algorithm to seek the solution of (30) and do not guarantee to find the global optimum of (30). Consequently, even if the sequences of iterates $\{S_i^{(\nu)}\}_{\nu=0}^{\infty}$ converge this does not by itself prove that an equilibrium has been found. We do not perform the necessary global optimization checks to verify that an equilibrium has been found.

As we argued in section 7.2, because of the integrated profit function this issue may be less problematic in the supply function space than it appears at first. This is because the non-concavity shown in the two firm example system in section 7.2 involved a supply bid by firm 2 that was extreme in that it became nearly flat at high prices. If a good initial guess of the solution of (30) can be used, such as a known equilibrium of a similar problem, then the low profit regions such as $S_1(19), S_1(40) \approx 0$ in the example in section 7.2 can be avoided.

All software was implemented using Matlab and the Matlab Optimization Toolbox.

9 Three firm numerical results

We used the symmetric three firm example to illustrate the results on stability of equilibria from section 5. In the following section we discuss the demand, price cap and price minimum, the supply functions, the starting functions, and the results.

9.1 Demand

We assumed a demand slope of $\gamma = 0.125$ GW per (pound per MWh) and a base-case load duration characteristic of:

$$\forall t \in [0,1], N(t) = 7 + 20(1-t),$$

with quantities measured in GW. That is, N varied linearly from 27 to 7 GW.

9.2 Price cap and price minimum

A price cap of $\overline{p} = 20$ pounds per MWh and a price minimum of $\underline{p} = 9$ pounds per MWh was used.

9.3 Supply functions

We used 40 break-points for most cases, with 20 break-points used to test the sensitivity of the results on the number of break-points.

9.4 Starting functions

In the case of symmetric cost functions and no capacity constraints nor price caps, we have already exhibited the range of equilibria between $S^{\star\text{Cournot}}$ and $S^{\star\text{comp}}$. We used a range of

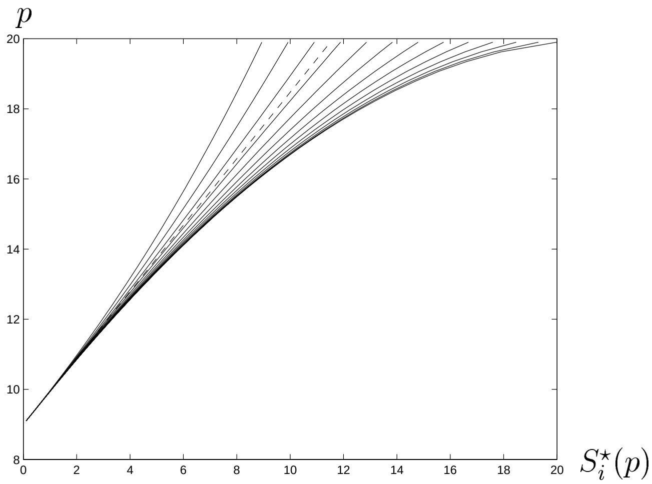

Figure 14: Starting functions for symmetric three firm example.

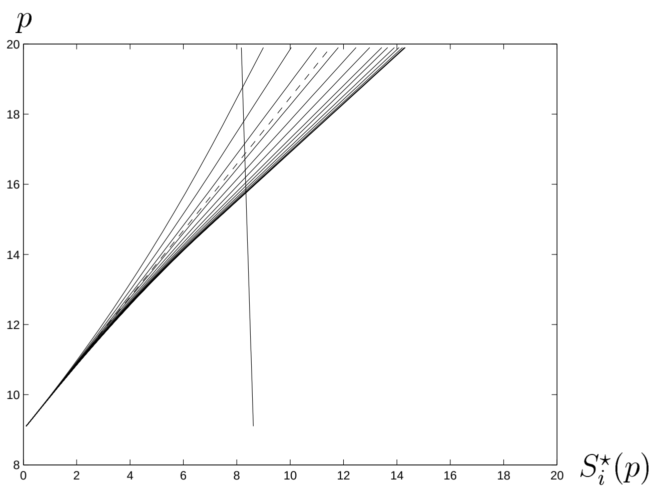

Figure 15: Perturbed starting functions constructed according to definition 4.

such equilibria as starting functions. We calculated the equilibria using (16) and included SFEs that were more competitive and also SFEs that were less competitive than the affine SFE $S^{\star\text{affine}}$. The starting functions are illustrated in figure 14. The affine SFE is shown dashed, while the others are shown solid. Since each SFE is symmetric, each supply function illustrated represents the supply functions of all three firms for that equilibrium.

We also used the construction in definition 4, with $p^\epsilon \approx p_0^\star - 1$ pound per MWh, to perturb the SFEs slightly. These perturbed SFEs are shown in figure 15. The nearly vertical line shows the vicinity of the peak realized prices for these supply functions.

9.5 Results using SFEs as starting functions

The results of using the SFEs as starting functions are shown in figure 16. The figure shows profits versus iteration for one of the firms (the profits are identical for each firm) for each of the starting functions. The case of the affine SFE is shown dashed. In every case, the profits are identical at each iteration. This shows that the numerical framework evaluates the profits correctly for these starting functions.

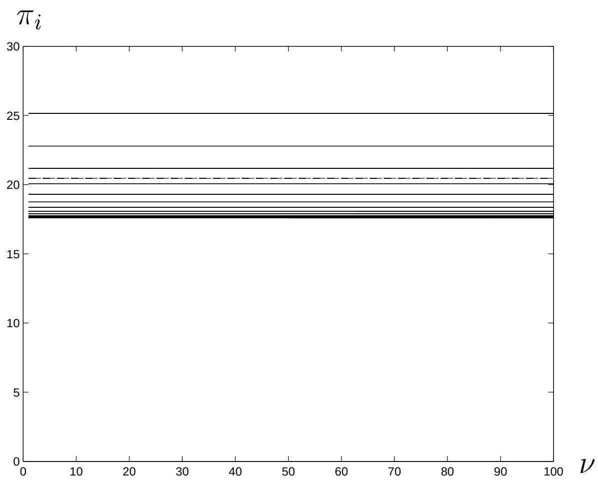

Figure 16: Profits versus iteration for SFE starting functions.

9.6 Results using perturbed SFEs as starting functions

The results of using the perturbed SFEs as starting functions are shown in figure 17. The figure shows profits versus iteration for a firm for each of the starting functions. The results are very different to those shown in figure 16. In particular, except for the affine SFE and the two SFEs either side of it, the sequence of profits differs significantly from the starting profits. For all but these three starting functions, the sequence of profits appears to be drifting towards a band of profits that is lower than the profits for the affine SFE. This result is, however, dependent on the details of the numerical calculation. For example, figure 18 shows the results using the similar starting functions but only 20 break-points in the functions. The sequence of profits is rather different.

By corollary 8, all SFEs produced according to (16) except the affine SFE are unstable. However, from a numerical perspective, it is not surprising that the SFEs that are "close" to the affine SFE appear to be stable on the basis of numerical calculations. Interestingly, the numerical results seem to also suggest that there is a band of stability involving SFEs that yield lower profits than the affine SFE. This may be an artifact of the use of piece-wise linear approximations to the functions, since the band seems to be dependent, for example, on the number of break-points.

10 Simulation assumptions for five firm example

In the following sections we discuss the costs and capacities, the price cap and price minimum, the starting functions, and the stopping criterion for assessing whether or not there are multiple equilibria.

10.1 Cost functions and capacities

We again consider the five firm example first introduced in section 6.3. The cost data is as in table 2. We also used the capacities as presented in [8] for the five strategic firm industry

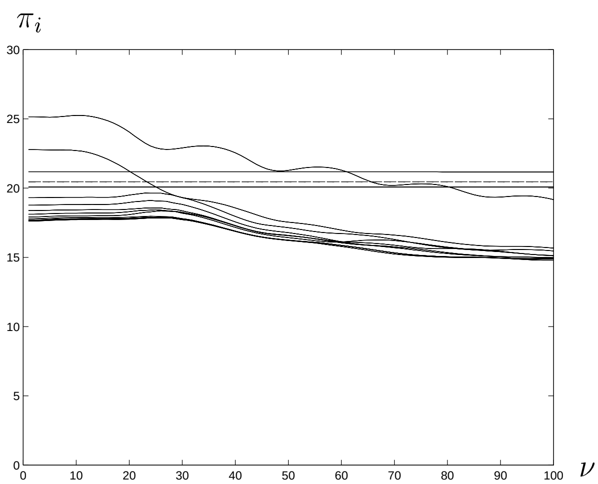

Figure 17: Profits versus iteration for perturbed SFE starting functions.

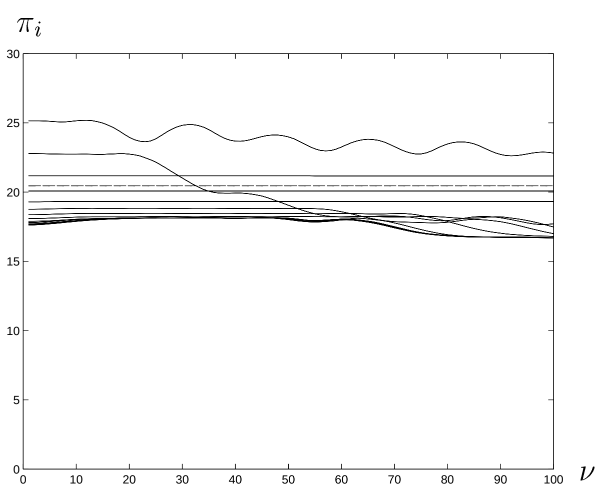

Figure 18: Profits versus iteration for perturbed SFE starting functions, with 20 break-points in supply functions.

Firm $i =$	1	2	3	4	5
$\overline{q}_i$(GW) $=$	5.70945	3.35325	10.4482	9.70785	3.3609

Table 3: Capacity data based on five firm industry described in [8].

in England and Wales subsequent to the 1999 divestiture. Table 3 shows the maximum capacities. The total installed capacity is approximately 32.6 GW and the marginal cost at maximum production is roughly 27 pounds per MWh for all firms. Firms 2 and 5 are nearly identical and have the smallest capacity. Firms 3 and 4 have the largest capacity.

10.2 Demand

We assumed a demand slope of $\gamma = 0.1$ GW per (pound per MWh) and a base-case load duration characteristic of:

$$\forall t \in [0, 1], N(t) = 10 + 25(1 - t),$$

with quantities measured in GW. That is, N varied linearly from 35 to 10 GW. This load-duration characteristic is illustrated in figure 1. The load factor is approximately 30%. (This is considerably smaller than a typical daily load factor. However, the five firms that we model from England and Wales do not include the baseload nuclear generation, so that the N we use is actually a residual after baseload is subtracted. Alternatively, we can imagine that there has been some forward contracting of baseload capacity [20].)

At a demand of 30 GW and a price of 30 pounds per MWh, the price elasticity of demand is 0.1. The "choke price" at peak is $N(0)/\gamma = 350$ pounds per MWh, while the "choke price" at minimum demand is $N(1)/\gamma = 100$ pounds per MWh.

Since the maximum capacities of the five firms sums to approximately 32.6 GW and the price cap was 30 pounds per MWh or above, there is enough capacity to meet the peak demand at a price that is below the price cap. For the price cap of 30 pounds per MWh, the peak demand can only just be met. For price caps up to approximately 60 pounds per MWh, each firm is "pivotal" in that if any firm withdrew all its capacity from the market then the price would rise to the price cap at some times around peak demand and non-economic rationing would result.

As sensitivity cases, we also considered N varying linearly from:

1. 35 to 20 GW,
2. 20 to 10 GW,
3. 40 to 10 GW, and
4. 10 to 1 GW.

The first and second sensitivity cases divide the base case time horizon into peak (35–20 GW) and off-peak (20–10 GW) times. Combining the results from both allows an evaluation of how the load factor affects the equilibrium profits and prices. The third sensitivity case requires demand rationing even with a price cap of 50 pounds per MWh. The last sensitivity case investigates when minimum capacity constraints are binding on some of the firms.

The assumption of a linear load-duration characteristic is not realistic, but simplifies the computational implementation because N can be inverted easily. By corollary 13, the set of SFEs depends on the range of N but not on its detailed functional form and so the candidate equilibria we obtain could also be used to estimate profits with a more realistic load-duration characteristic. Nevertheless, the assumption of a linear N may affect which equilibrium is exhibited by the numerical framework if there are multiple equilibria.

10.3 Price cap and price minimum

A price cap of $\overline{p} = 40$ pounds per MWh was used as the base-case price cap. Since the maximum marginal cost of production is approximately 27 pounds per MWh, the base case price cap is nearly 50% higher than the maximum marginal production cost. Sensitivity cases with price caps in the range of 30–80 pounds per MWh were also considered. We also considered the case of bid caps at $\overline{p} = 40$ pounds per MWh.

For the cases with $N(1) \geq 10$, even competitive bids by all the players would result in prices above 12 pounds per MWh. A price minimum of $\underline{p} = 12$ pounds was used for most of these cases. A sensitivity case using $\underline{p} = 8$ pounds per MWh was used to verify that the choice of $\underline{p}$ did not tangibly affect results. The price minimum of $\underline{p} = 8$ pounds per MWh was also used for the cases with $N(1) = 1$ GW.

10.4 Starting functions

In the case of symmetric cost functions and no capacity constraints nor price caps, we have already exhibited a range of equilibria, including the three equilibria: $S^{\star\text{Cournot}}$, $S^{\star\text{affine}}$, and $S^{\star\text{comp}}$. Unfortunately, for the asymmetric cost functions we consider, supply functions $S^{\star\text{Cournot}}$ and $S^{\star\text{comp}}$ constructed using (16) with Cournot and competitive initial conditions, respectively, both violate the non-decreasing constraints for prices below the peak realized price. The functions $S_i^{\star\text{Cournot}}, i = 1, \ldots, n$ are illustrated in figure 19 and violate the non-decreasing constraints for prices less than about 64 pounds per MWh. (We continue to use a superscript $\star$ for these functions, although they are not even allowable supply functions if demand results in prices being realized on which the functions are not non-decreasing.) At a price of 64 pounds per MWh, the supply is approximately 22 GW. This corresponds to a value on the load-duration characteristic of $N(t) = 28$. That is, $S_i^{\star\text{Cournot}}$ could be an SFE for a system with load-duration characteristic that had range $[28, 35]$, which is a load factor of about 80%. (We note that even in this case, $S_i^{\star\text{Cournot}}$ is concave for firms 1, 3, 4, so that the equilibrium may be unstable.) For load factors below 80%, as in our example cases, $S_i^{\star\text{Cournot}}$ violates the non-decreasing constraints over the range of realized prices and therefore is not an equilibrium for such load-duration characteristics.

Were were unable to solve the differential equations starting from the competitive initial

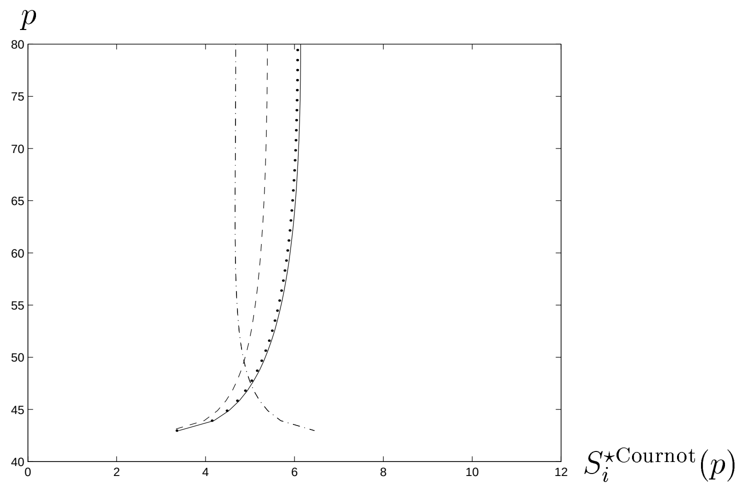

Figure 19: Supply function $S^{\star\text{Cournot}}$. Firm 1 is shown as a dashed line, firms 2 and 5 are shown superimposed as a dash-dot line, firm 3 is shown as a solid line and firm 4 is shown as a dotted line.

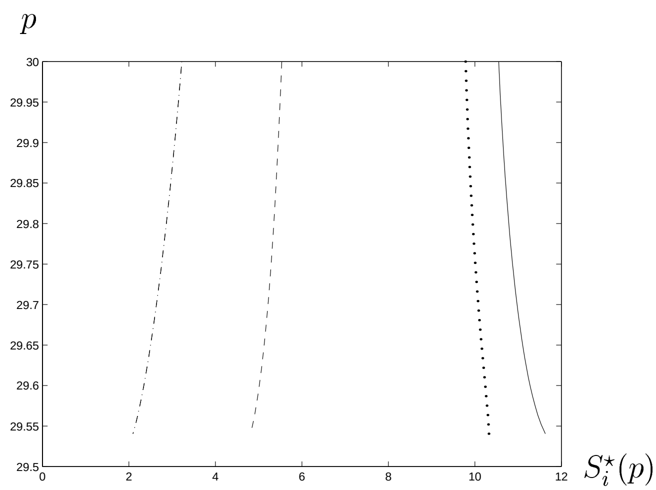

Figure 20: Supply function obtained using initial condition nearby to competitive initial condition. Firm 1 is shown as a dashed line, firms 2 and 5 are shown superimposed as a dash-dot line, firm 3 is shown as a solid line and firm 4 is shown as a dotted line.

condition to obtain $S^{\star\text{comp}}$. However,

$$\lim_{p \to p_0^{\text{comp}-}} S^{\star\text{comp}\prime}(p) = \begin{bmatrix} -\infty \\ -\infty \\ +\infty \\ +\infty \\ -\infty \end{bmatrix},$$

where $p \to p_0^{\text{comp}-}$ means the limit from below. That is, $S^{\star\text{comp}}$ must also violate the non-decreasing constraints. We were able to solve the differential equations for initial conditions nearby to the competitive initial condition. One such solution is illustrated in figure 20. All such nearby solutions violate the non-decreasing constraints.

The function $S^{\star\text{affine}}$ is well-defined in both the symmetric and asymmetric cases and we use it as a starting function. However, since $S^{\star\text{Cournot}}$ and $S^{\star\text{comp}}$ are not allowable functions, we defined two other starting functions, one less and the other more competitive than the affine SFE $S^{\star\text{affine}}$. In particular, for the unconstrained and no price cap case we used three different starting functions:

- "uncapacitated competitive," S^{comp} where the supply functions are the inverses of the marginal cost functions,
- "uncapacitated affine SFE," $S^{\star\text{affine}}$ where the supply functions are given by the solution of the affine SFE (14), with coefficients β_i satisfying (15), and
- "uncapacitated Cournot," S^{Cournot} where quantities and prices under Cournot competition are calculated for each $t \in [0, 1]$ and a supply function drawn through the resulting price-quantity pairs.

For the maximum capacity constrained and price-capped cases, we used the following three starting functions:

- "capacitated competitive," where the supply functions are the inverses of the marginal cost functions, but limited by the maximum capacity, as shown in figure 21,
- "capacitated affine SFE," where the supply functions are given by the solution of the affine SFE, except that the values of S_i are limited by the maximum capacity, as shown in figure 22, and
- "price-capped Cournot," where Cournot quantities and prices are calculated for each $t \in [0, 1]$ and a supply function drawn through the resulting price-quantity pairs, but modified to satisfy (4), as shown in figure 23.

(In each case, we have graphed the supply function only for prices greater than 12 pounds per MWh, to avoid the issue of minimum capacity constraints under the assumption that realized prices are always at least 12 pounds per MWh.) We will discuss this issue, and provide a generalization of $S^{\star\text{affine}}$ when there are minimum capacity constraints in section 11.2.) In summary, the starting functions for the capacitated and price-capped cases are obtained by calculating a supply curve under the assumption of no capacity constraints and then truncating the supply curve to satisfy the capacity constraints and then (in the case of price-capped Cournot) redefining the supply function at the price $p = \overline{p}$ so that it satisfies (4).

Firms 2 and 5 are essentially identical and their supply functions appear superimposed as the leftmost dash-dot curve in figures 21–23 and in all subsequent figures. Firms 3 and 4 have the largest capacity and their supply functions appear as the solid and dotted curves, respectively, at the right of figures 21–23 and in all subsequent figures. (In figure 23, the supply functions of firms 3 and 4 are almost superimposed.) The supply function of firm 1 appears as the dashed curve in the middle of figures 21–23 and in all subsequent figures.

Although the starting functions are not equilibrium supply functions for the capacitated and price-capped cases, we can still consider the resulting prices if the firms were to bid these supply functions. The price-duration curves for the base case demand conditions corresponding to the firms bidding the capacitated competitive, the capacitated affine SFE, and the price-capped Cournot supply functions, respectively, are shown in figures 24–26, respectively. Given bids equal to the capacitated competitive supply function, no firm ever reaches its capacity and so the price-duration curve in figure 24 is linear.

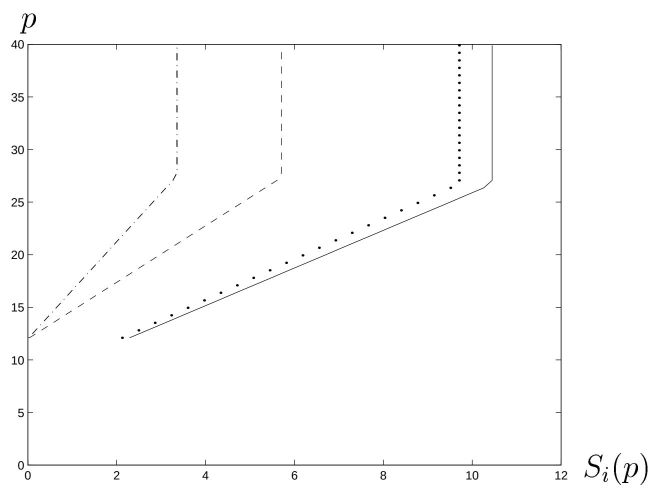

Figure 21: “Capacitated competitive” supply function.

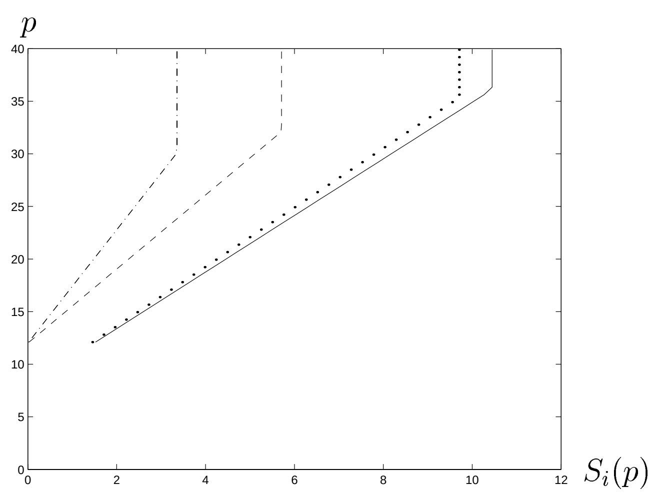

Figure 22: “Capacitated affine SFE” supply function.

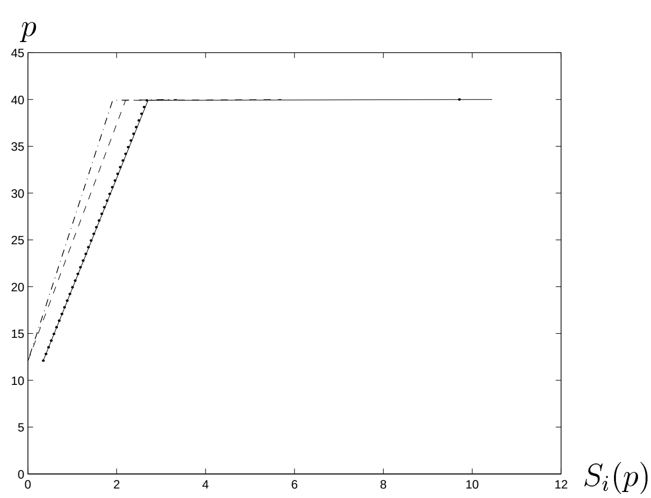

Figure 23: “Price-capped Cournot” supply function. (Note change in price axis compared to previous figures.)

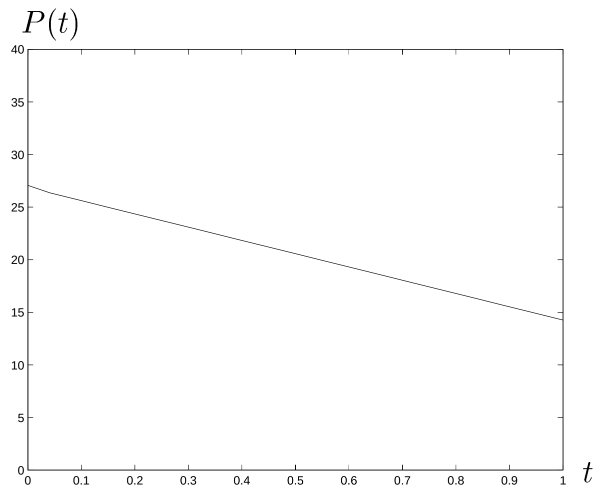

Figure 24: Price-duration curve for "capacitated competitive" supply function.

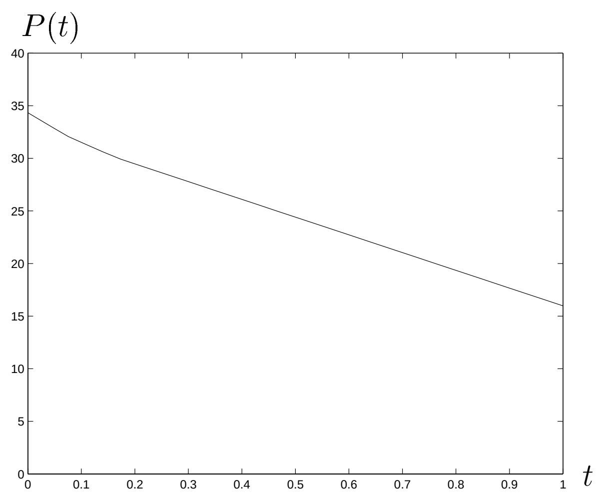

Figure 25: Price-duration curve for "capacitated affine SFE" supply function.

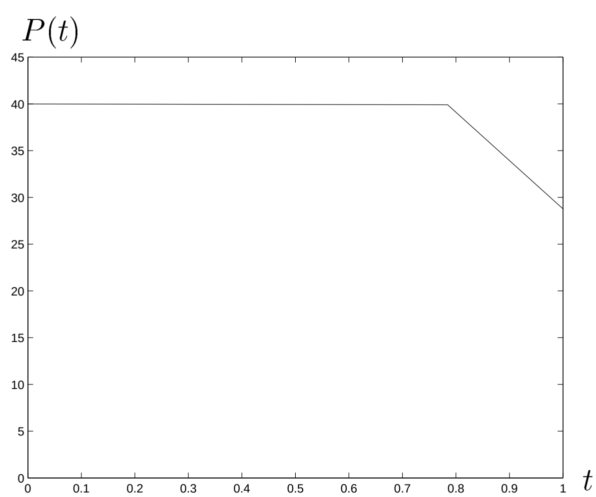

Figure 26: Price-duration curve for "price-capped Cournot" supply function. (Note change in price axis compared to previous figures.)

Given bids equal to the capacitated affine SFE supply functions, capacity constraints are reached for firms 2 and 5 at a price of about 30 pounds per MWh, so that the price-duration curve in figure 25 bends upward for peak demand times near to $t = 0$. A reasonable hypothesis is that the capacitated affine SFE starting function is in the vicinity of the equilibrium for the base case since it is the equilibrium if the capacity constraints are not binding.

Given bids equal to the price-capped Cournot supply function, no firm ever reaches its capacity. However, the price cap is binding over most of the time horizon as shown in figure 26. As suggested in the introduction, the prices and profits corresponding to the price-capped Cournot supply function may be a reasonable prediction of the equilibrium behavior when firms face the price cap but are not required to bid consistently across the time horizon or if the load-duration characteristic is piece-wise constant. We will use these "price-capped Cournot" prices and profits as a benchmark to evaluate the effect of requiring supply function bids that are consistent over the time horizon. (For comparison, the Cournot price corresponding to the peak time and with no price cap is around 80 pounds per MWh.)

10.5 Criterion for assessing existence of multiple equilibria

In experiments, we found that even after a large number of iterations, the values of the supply functions were still changing by significant amounts from iteration to iteration. In particular, the L_1 norm of the difference between successive iterates was on the order of a few percent of the L_1 norm of the iterate itself. Moreover, supply functions change visibly from iteration to iteration, with the position of features such as points of non-differentiability in the supply functions slowly drifting over successive iterations.

In contrast, profit at each iteration showed much steadier progress. Defining profit at iteration ν according to (8) with the supply functions $S_i^{(\nu)}$ used to specify the price function through (6), we found that the profits typically changed by less than 0.1% from iteration to iteration after 100 iterations. Moreover, the profits typically reach a quasi-steady state level within about 20 iterations.

As suggested by [9], the changes in bid functions from iteration to iteration may be evidence of Edgeworth cycles. However, the steadiness of the profit functions suggests that the changes in the supply functions may simply be an artifact of the numerical calculations.

In assessing whether or not there are multiple equilibria, we must distinguish differences due to artifacts of the calculations from truly different equilibria. We apply the following *ad hoc* criterion. We deem two candidate equilibria to be the same if:

- for each firm, the profits are within 2% in each candidate equilibrium,
- for each firm, the supply functions have the same general shape in each candidate equilibrium over the range of realized prices, and
- the price-duration curves have the same general shape in each candidate equilibrium (and, in particular, have the same peak realized price.)

For each case, we iterated 100 times from the starting function and used the results from iteration 100 to assess whether or not candidate equilibria were the same or different.

When multiple equilibria are observed, we consider the range of equilibrium outcomes. In assessing whether the range of profits is relatively large or small, we compare the range of profits for the various equilibria to the range between:

- the profits that would accrue if all firms bid the capacitated competitive supply function, shown in figure 21 and
- the profits that would accrue if all firms bid the price-capped Cournot supply function, shown in figure 23.

That is, the difference between the competitive and Cournot profits provides a scale for assessing the relative spread of profits when there are multiple equilibria.

11 Five firm numerical results

In this section, we report results of several cases:

- no capacity constraints,
- minimum capacity constraints,
- base case demand and supply conditions,
- changed price caps,
- increased capacities,
- increased load factor,
- increased demand.

We investigate empirically the conditions for the results to exhibit multiple equilibria and also the qualitative effects of the changes compared to the base case.

11.1 No capacity constraints

If market rules require that an affine supply function be bid by each firm, then in the case of no capacity constraints the affine SFE $S^{\star\text{affine}}$ is the unique SFE. If market rules allow nonlinear supply functions, then in the case of no capacity constraints there is a continuum of supply function equilibria, with the affine solution $S^{\star\text{affine}}$ being one of them.

We used the software to solve the no capacity constraints, no price cap, and nonlinear bid supply function case for the base case demand. We used starting functions equal to, respectively:

- the uncapacitated competitive supply function, S^{comp},
- the uncapacitated affine SFE supply function $S^{\star\text{affine}}$, and
- the uncapacitated Cournot supply function, S^{Cournot}.

The test run serves to verify the operation of the software on a problem for which we know one of the equilibria, namely the affine SFE. Using the affine SFE as a starting function serves to verify that the software evaluates the profit correctly.

11.1.1 Uncapacitated competitive starting function

Figure 27 shows the profits versus iteration ν for the no capacity limit case starting from the uncapacitated competitive supply function. (In this figure and all subsequent figures illustrating the five firm example, firm 1 is shown as a dashed line, firms 2 and 5 have identical costs and capacities and are shown superimposed as a dash-dot line, firm 3 is shown as a solid line and firm 4 is shown as a dotted line.) The leftmost points in figure 27 show the profits if each firm were to bid the uncapacitated competitive supply function. That is, these are the profits if each firm bid competitively.

Figure 28 shows the corresponding supply functions at iteration 100. The price-duration curve for iteration 100 is shown in figure 29. The peak realized price is 29 pounds per MWh.

11.1.2 Uncapacitated affine SFE starting function

Figure 30 shows the profits versus iteration ν for the no capacity limit case starting from the uncapacitated affine SFE supply function. Profits are identical in every iteration. The leftmost points in figure 30 show the profits if each firm were to bid the uncapacitated affine SFE supply function. That is, these are the equilibrium profits if the firms are required to bid affine supply functions.

Figure 31 shows the corresponding supply functions at iteration 100, which are identical to the uncapacitated affine SFE. The price-duration curve for iteration 100 is shown in figure 32. The peak realized price is between 32 and 33 pounds per MWh.

11.1.3 Uncapacitated Cournot starting function

Figure 33 shows the profits versus iteration ν for the no capacity limit case starting from the uncapacitated Cournot supply function. The leftmost points in figure 33 show the profits if each firm were to bid the uncapacitated Cournot supply function. That is, these are the profits if Cournot competition occurs at each time in the time horizon without any obligation to bid a supply function that is consistent across the whole time horizon.

Figure 34 shows the corresponding supply functions at iteration 100. The price-duration curve for iteration 100 is shown in figure 35. The peak realized price is again between 32 and 33 pounds per MWh.

11.1.4 Summary

From the perspectives of:

- the profit;
- the shape of the supply functions over the range of realized prices; and,
- the price-duration curves,

the results at iteration 100 starting from the uncapacitated affine SFE $S^{\star\text{affine}}$ and the uncapacitated Cournot functions are very similar. However, these two results differ from the results at iteration 100 starting from the uncapacitated competitive supply function. In

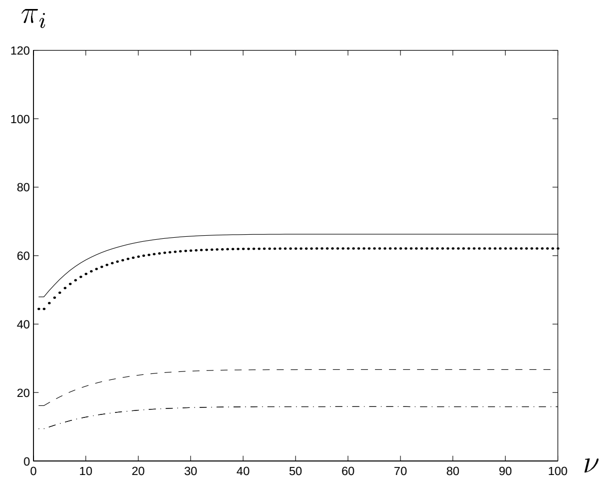

Figure 27: Profits versus iteration for case of no capacity constraints, starting from the uncapacitated competitive supply function.

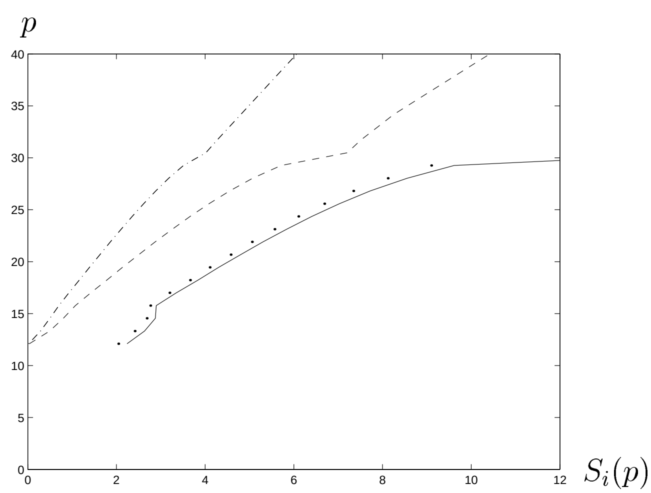

Figure 28: Supply functions at iteration 100 for case of no capacity constraints, starting from the uncapacitated competitive supply function.

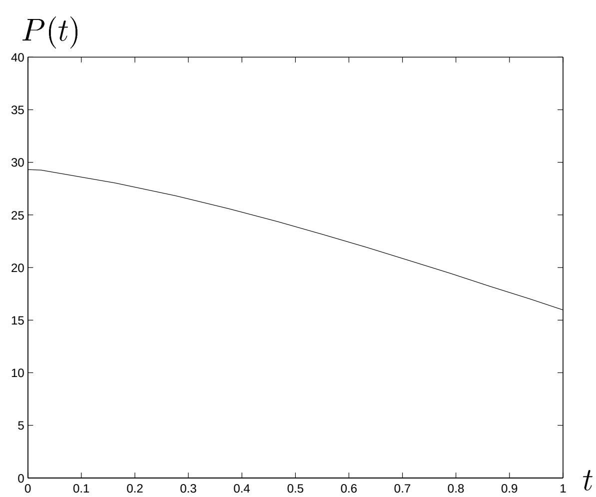

Figure 29: Price-duration curve at iteration 100 for case of no capacity constraints, starting from the uncapacitated competitive supply function.

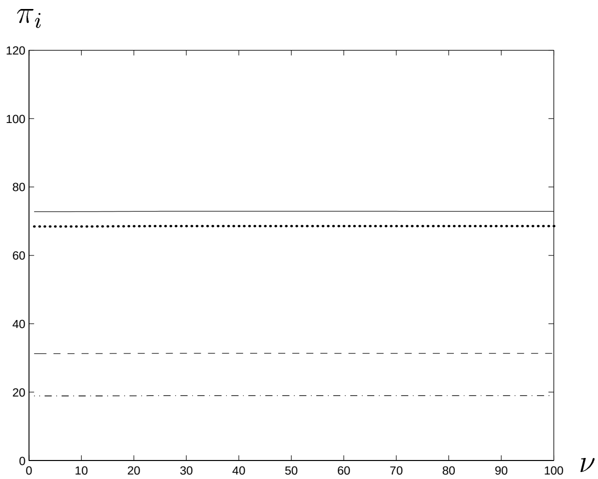

Figure 30: Profits versus iteration for case of no capacity constraints, starting from the uncapacitated affine SFE supply function.

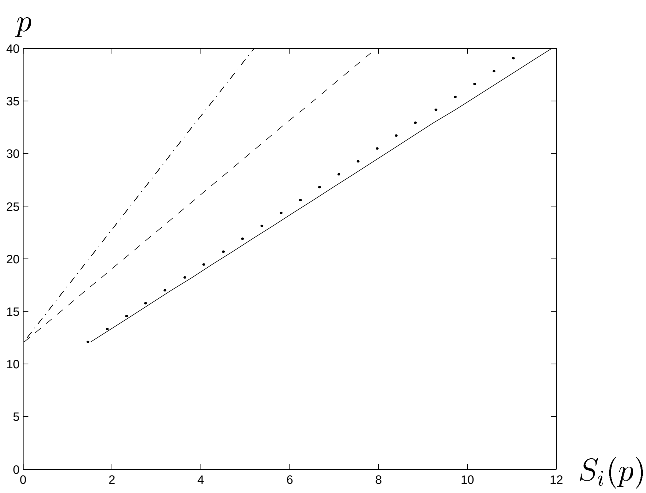

Figure 31: Supply functions at iteration 100 for case of no capacity constraints, starting from the uncapacitated affine SFE supply function.

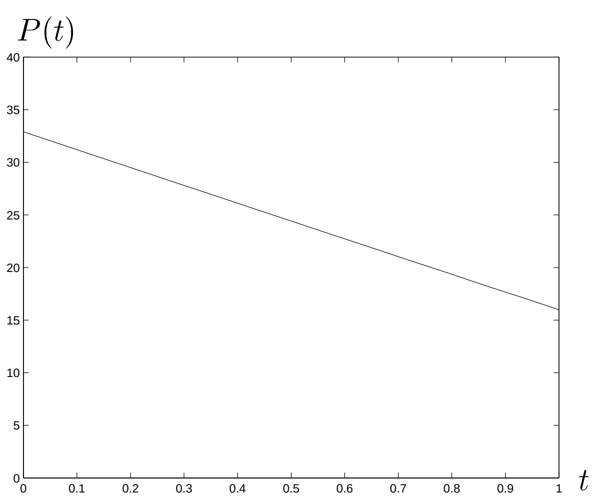

Figure 32: Price-duration curve at iteration 100 for case of no capacity constraints, starting from the uncapacitated affine SFE supply function.

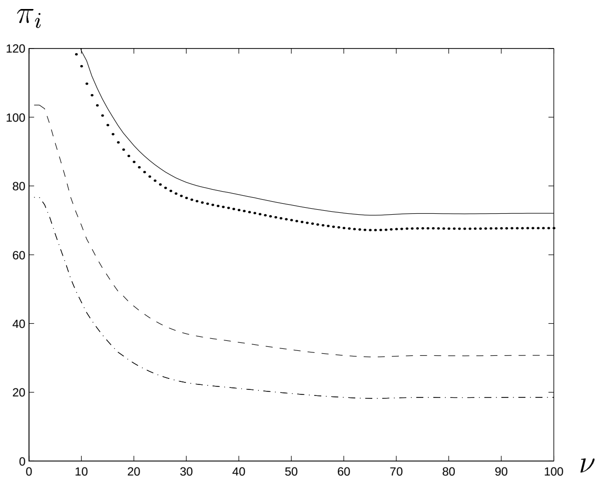

Figure 33: Profits versus iteration for case of no capacity constraints, starting from the uncapacitated Cournot supply function.

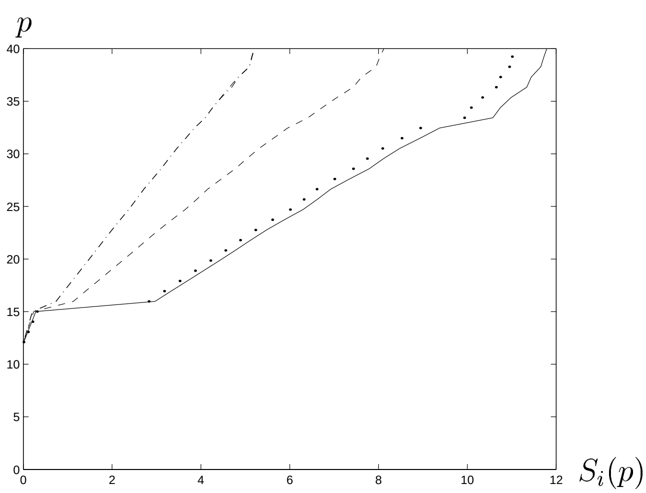

Figure 34: Supply functions at iteration 100 for case of no capacity constraints, starting from the uncapacitated Cournot supply function.

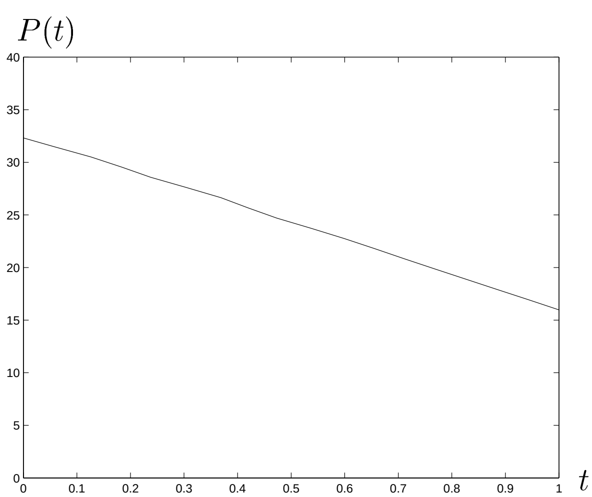

Figure 35: Price-duration curve at iteration 100 for case of no capacity constraints, starting from the uncapacitated Cournot supply function.

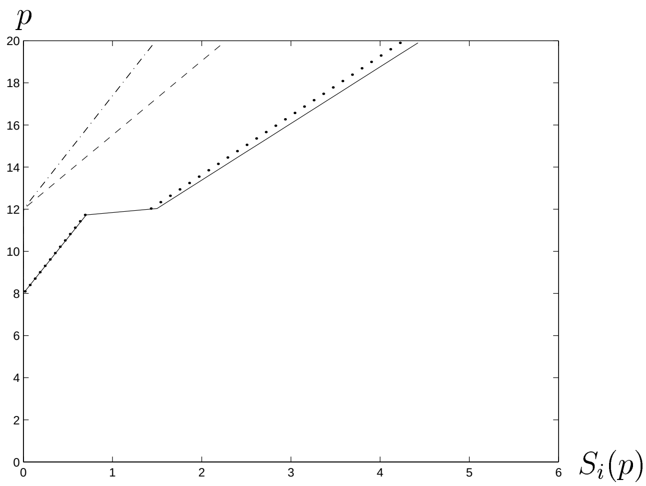

Figure 36: Piece-wise affine SFE constructed according to [8].

particular, compared to the results at iteration 100 starting from the affine SFE or Cournot supply functions:

- the profits at iteration 100 starting from the uncapacitated competitive starting function are about 15% lower and
- the values at iteration 100 of $S_i(p)$ starting from the uncapacitated competitive starting function are considerably higher for prices above about 20 pounds per MWh.

The numerical results at iteration 100 show two candidate equilibria and there may be a continuum of equilibria between these two. We consider the relative range of the profits for the two candidate equilibria. For firm 1, for example, the range of profits at iteration 100 over the various start functions is from about 27 to 32, a range of 5.

The profits that would accrue to firm 1 if all firms bid the uncapacitated Cournot supply function are about 104. The profits that would accrue to firm 1 if all firms bid the uncapacitated competitive supply function are about 16. This is a range of about 88.

Combining these observations, the range of profits at iteration 100 for firm 1 over the various start functions is only about 6% of the range of profits for firm 1 between uncapacitated competitive and uncapacitated Cournot outcomes. That is, the range of SFE profits is relatively small. Similar observations apply for the other firms. Again, the range of apparently stable equilibria may be an artifact of the numerical framework.

11.2 Minimum capacity constraints

In this section, we use a reduced demand with $N(0) = 10$, $N(1) = 1$ in order to investigate the effects of *minimum* capacity constraints during off-peak times. In [8], piece-wise affine (but not continuous) SFEs are exhibited in the case of minimum capacity constraints. In this SFE, the equilibrium supply function of a firm i is discontinuous at any price p where a rival $j \neq i$ has cost function satisfying $a_j = p$. Using the results from [8] for the five firm example system results in the supply functions shown in figure 36.

We used the software to solve the minimum capacity constraints, no price cap, and nonlinear bid supply function case for the demand specified by $N(0) = 10$, $N(1) = 1$. Because the supply functions shown in figure 36 are an equilibrium in piece-wise affine functions, we used this as one of the starting functions (and will refer to it as $S^{\star\text{affine}}$.) Since we use a piece-wise linear and continuous representation of functions, we can only approximate the jump in this function at $p = 12$ pounds per MWh. We also used the competitive and Cournot starting functions and represented the minimum capacity limits in these functions by requiring the functions to be non-negative.

11.2.1 Competitive starting function

Figure 37 shows the profits versus iteration ν for the minimum capacity limit case starting from the uncapacitated competitive supply function. The leftmost points in figure 37 show the profits if each firm were to bid competitively. (The axes of the graphs in this section differ from that in section 11.1.)

Figure 38 shows the corresponding supply functions at iteration 100. The price-duration curve for iteration 100 is shown in figure 39. The peak realized price is 15 pounds per MWh.

11.2.2 Piece-wise affine SFE starting function

Figure 40 shows the profits versus iteration ν for the minimum capacity limit case starting from the uncapacitated affine SFE supply function. Profits are almost identical in every iteration.

Figure 41 shows the corresponding supply functions at iteration 100, which are similar to the piece-wise affine SFE $S^{\star\text{affine}}$, except that the discontinuity at $p = 12$ pounds per MWh in $S^{\star\text{affine}}$ is smoothed off in the numerical results at iteration 100. Figure 40 shows that the smoothing off had essentially no effect on the profits of the firms. The price-duration curve for iteration 100 is shown in figure 42. The peak realized price is about 16 pounds per MWh.

11.2.3 Cournot starting function

Figure 43 shows the profits versus iteration ν for the minimum capacity limit case starting from the Cournot supply function. The leftmost points in figure 43 show the profits if each firm were to bid the Cournot supply function. That is, these are the profits if Cournot competition occurs at each time in the time horizon without any obligation to bid a supply function that is consistent across the whole time horizon.

Figure 44 shows the corresponding supply functions at iteration 100. The price-duration curve for iteration 100 is shown in figure 45. The peak realized price is about 16 pounds per MWh.

11.2.4 Summary

From the perspectives of:

- the profit;
- the shape of the supply functions over the range of realized prices; and,

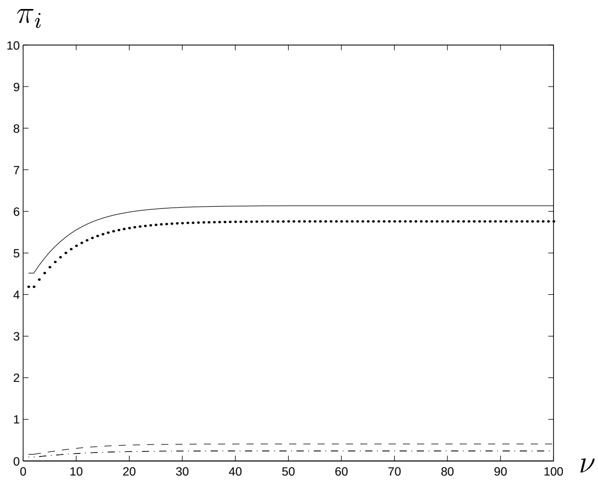

Figure 37: Profits versus iteration for case of minimum capacity constraints, starting from the competitive supply function.

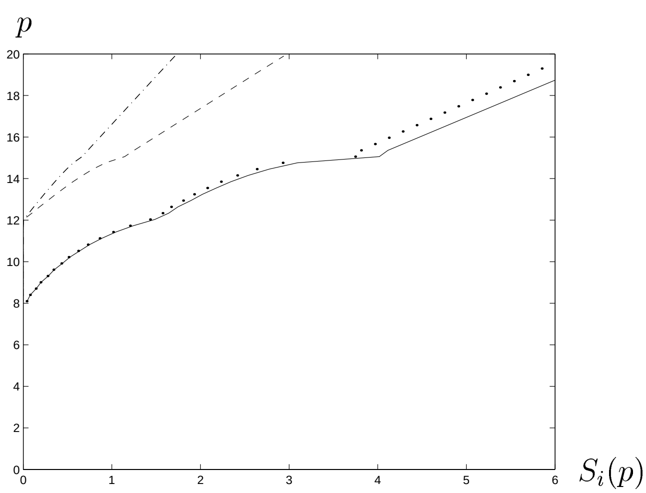

Figure 38: Supply functions at iteration 100 for case of minimum capacity constraints, starting from competitive supply function.

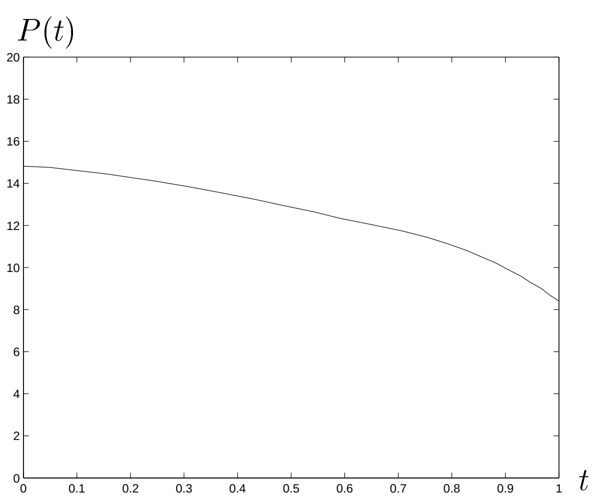

Figure 39: Price-duration curve at iteration 100 for case of minimum capacity constraints, starting from uncapacitated competitive supply function.

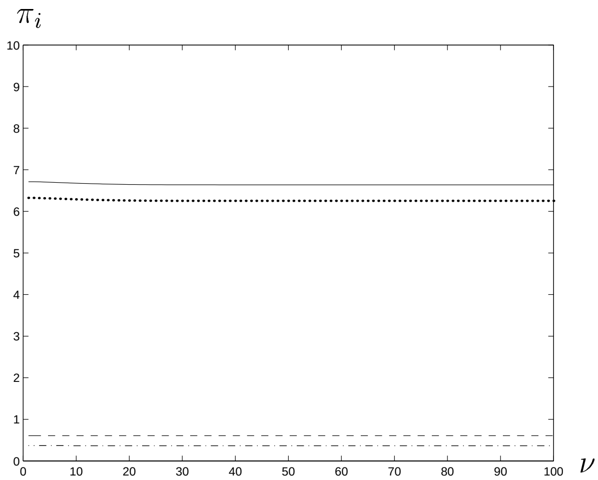

Figure 40: Profits versus iteration for case of minimum capacity constraints, starting from the piece-wise affine SFE supply function.

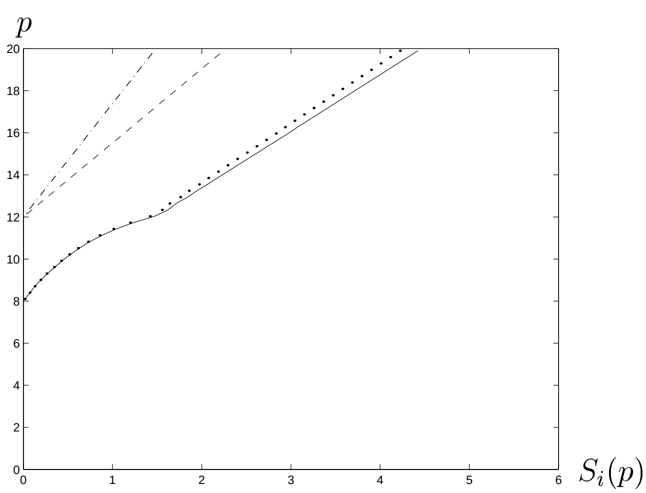

Figure 41: Supply functions at iteration 100 for case of minimum capacity constraints, starting from the piece-wise affine SFE supply function.

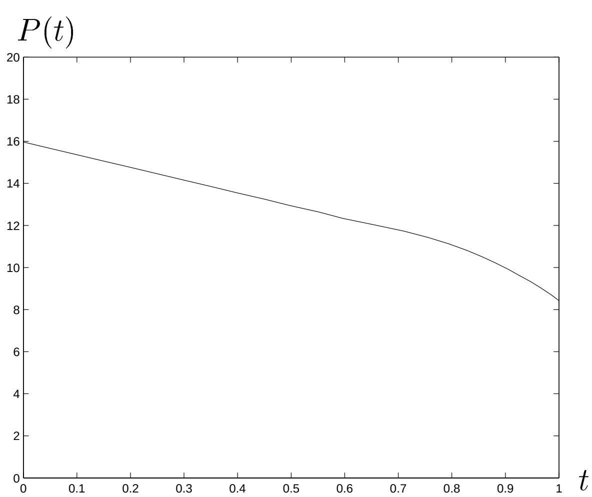

Figure 42: Price-duration curve at iteration 100 for case of minimum capacity constraints, starting from the piece-wise affine SFE supply function.

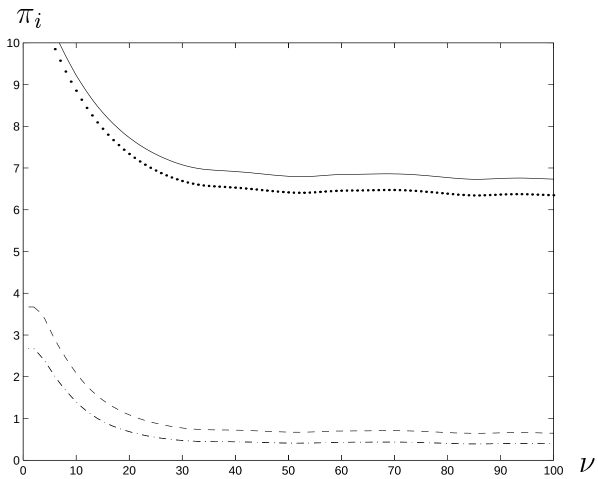

Figure 43: Profits versus iteration for case of minimum capacity constraints, starting from Cournot supply function.

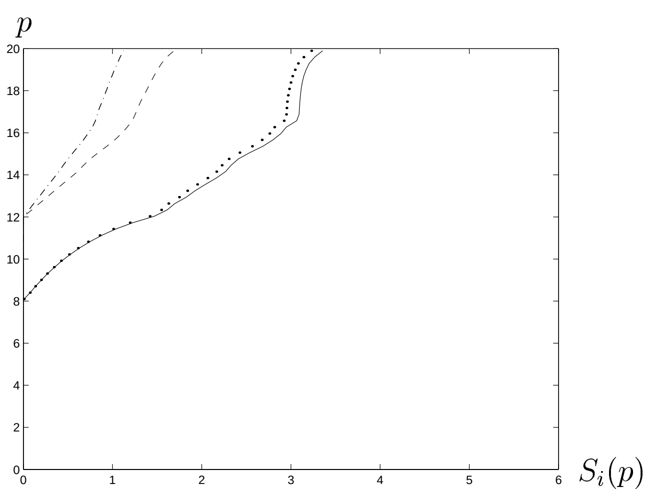

Figure 44: Supply functions at iteration 100 for case of minimum capacity constraints, starting from Cournot supply function.

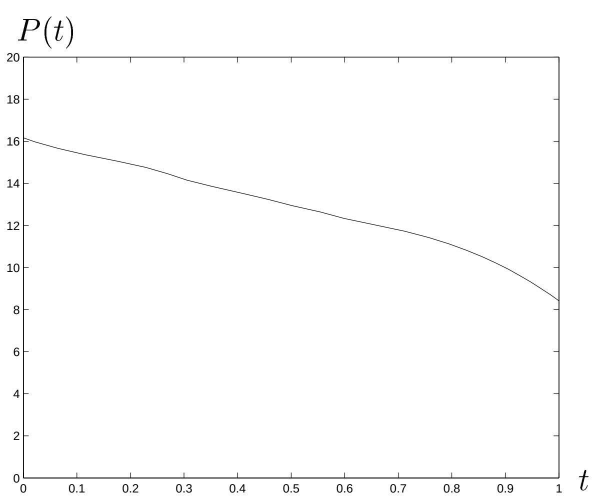

Figure 45: Price-duration curve at iteration 100 for case of minimum capacity constraints, starting from Cournot supply function.

- the price-duration curves,

the results at iteration 100 starting from the piece-wise affine SFE $S^{\star \text{affine}}$ and the Cournot functions are very similar. However, these two results differ from the results at iteration 100 starting from the competitive supply function. In particular, compared to the results at iteration 100 starting from the affine SFE or Cournot supply functions:

- the profits at iteration 100 starting from the uncapacitated competitive starting function are again somewhat lower and
- the values at iteration 100 of $S_i(p)$ starting from the uncapacitated competitive starting function are higher for prices above about 12 pounds per MWh.

The numerical results at iteration 100 again show two candidate equilibria. However, the range of SFE profits is again relatively small.

11.3 Base case

We used the software to seek the equilibrium for the base case assumptions, which involves capacity constraints and a price cap.

11.3.1 Starting from capacitated competitive

Figure 46 shows the profits versus iteration ν for the base case assumptions starting from the capacitated competitive supply function. The leftmost points in figure 46 show the profits if each firm were to bid the capacitated competitive supply function. That is, these are the profits if the firms bid competitively at all times. The price-duration curve if each firm were to bid the capacitated competitive supply function is shown in figure 24. (The axes of the graphs in this section differ from that in section 11.2, but are similar to that in section 11.1.)

Profits at iteration 100 are considerably higher than in the uncapacitated case and more than double the profits that would accrue if the capacitated competitive supply functions were bid. As previously, firms 2 and 5 have identical costs and capacities, so they appear superimposed as the dash-dot curve.

Figure 47 shows the supply functions at iteration 100. The price-duration curve for iteration 100 is shown in figure 48.

11.3.2 Starting from capacitated affine SFE

Figure 49 shows the profits versus iteration ν for the base case assumptions starting from the capacitated affine SFE supply function. The leftmost points in figure 49 show the profits if each firm were to bid the capacitated affine SFE supply function. The price-duration curve if each firm were to bid the capacitated affine SFE supply function is shown in figure 25.

Profits at iteration 100 are again considerably higher than if all firms bid the capacitated affine starting function.

Figure 50 shows the supply functions at iteration 100. The price-duration curve for iteration 100 is shown in figure 51. The results at iteration 100 starting from capacitated affine SFE are similar to the case of starting from the capacitated competitive supply function.

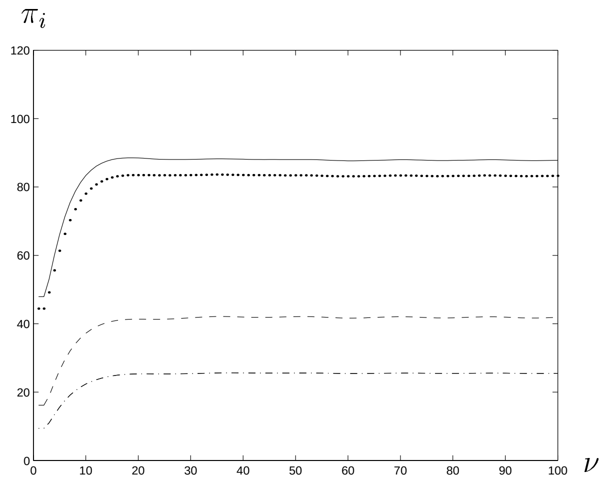

Figure 46: Profits versus iteration for base case assumptions starting from capacitated competitive.

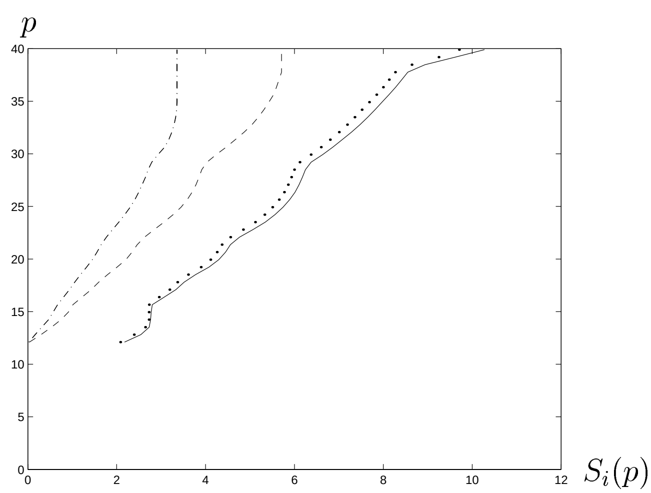

Figure 47: Supply functions at iteration 100 for base case assumptions starting from capacitated competitive.

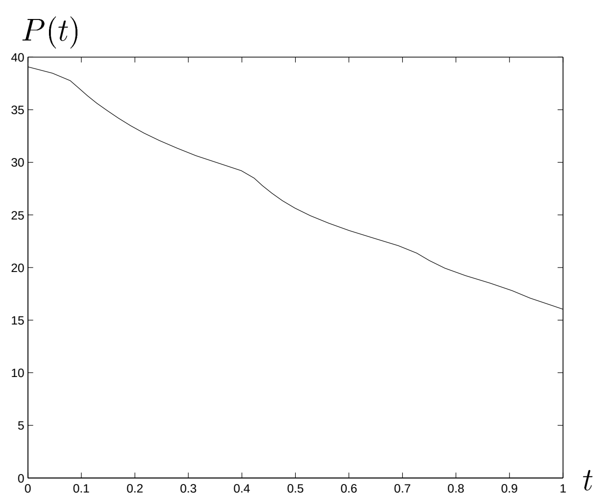

Figure 48: Price-duration curve at iteration 100 for base case assumptions starting from capacitated competitive.

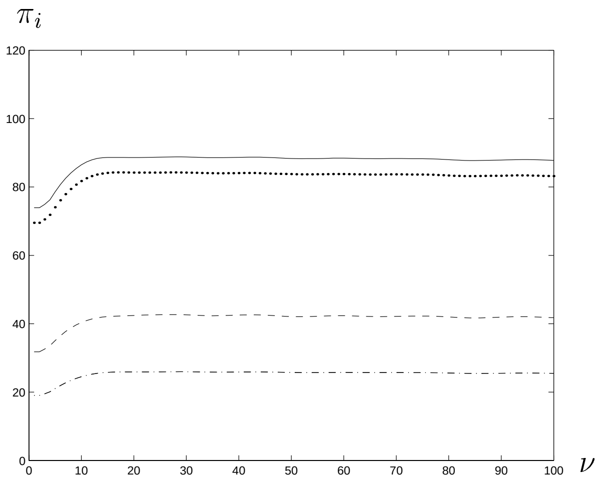

Figure 49: Profits versus iteration for base case assumptions starting from capacitated affine SFE.

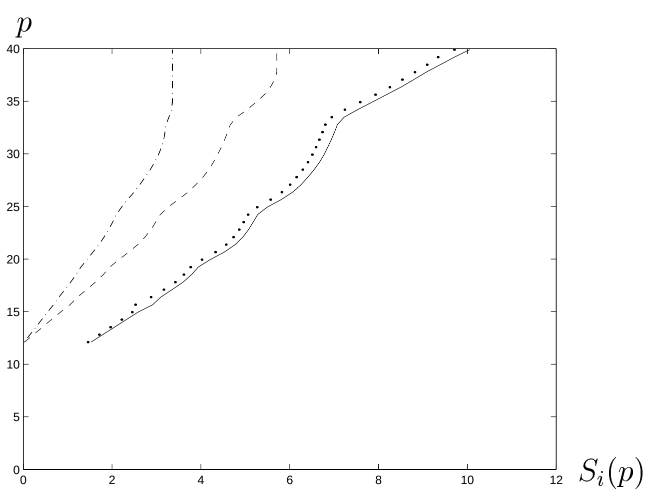

Figure 50: Supply functions at iteration 100 for base case assumptions starting from capacitated affine SFE.

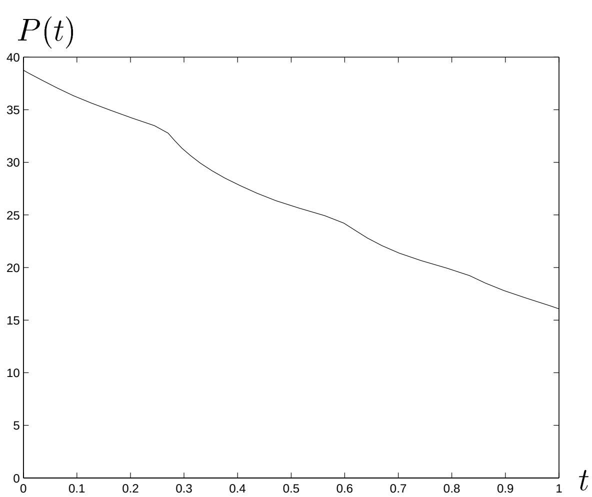

Figure 51: Price-duration curve at iteration 100 for base case assumptions starting from capacitated affine SFE.

11.3.3 Starting from price-capped Cournot

Figure 52 shows the profits versus iteration ν for the base case assumptions starting from the price-capped Cournot supply function. The leftmost points in figure 52 show the profits if each firm were to bid the price-capped Cournot supply function. That is, these are the equilibrium profits if price-capped Cournot competition occurs at each time in the time horizon without any obligation to bid a supply function that is consistent across the time horizon. The price-duration curve if each firm were to bid the price-capped Cournot supply function is shown in figure 26.

Profits at iteration 100 are considerably lower than if all firms bid the price-capped Cournot supply function.

Figure 53 shows the supply functions at iteration 100. The price-duration curve for iteration 100 is shown in figure 54. The supply curves differ significantly from the previous cases for prices less than 16 pounds per MWh; however, these prices are below the minimum realized price and so are not relevant in the calculation of profits.

11.3.4 Starting from price-capped Cournot with reduced price minimum

Figure 55 shows the profits versus iteration ν for the base case assumptions starting from the price-capped Cournot supply function, except that the price minimum was reduced to $\underline{p} = 8$ pounds per MWh. Figure 56 shows the supply functions at iteration 100. The price-duration curve for iteration 100 is shown in figure 57. The results are similar to figures 52–54 except that the price-duration curve is slightly different for prices between 25 and 35 pounds per MWh.

11.3.5 Reduced number of break-points

Figure 58 shows the profits versus iteration ν for the base case assumptions starting from the capacitated affine SFE supply function but with only 20 break-points in the supply function. Figure 59 shows the supply functions at iteration 100.

Figures 50 and 59 both show the results at iteration 100 starting from the capacitated affine SFE starting function. The difference is that figure 50 involves supply functions with 40 break-points while figure 59 involves supply functions with 20 break-points. The differences between the supply functions in these figures is an artifact of the numerical technique. The differences seem qualitatively to be of the same magnitude as the differences between these figures and the results at iteration 100 for the other starting functions. Consequently, we hypothesize that the differences in the supply functions at iteration 100 for the various starting functions are all artifacts of the numerical technique and not indicative of multiple equilibria.

The price-duration curve for iteration 100 starting from the capacitated affine SFE supply function with 20 break-points is shown in figure 60. The results at iteration 100 are slightly different from the previous results.

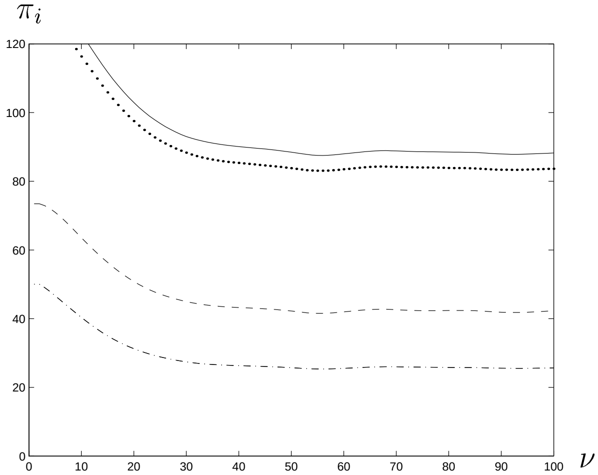

Figure 52: Profits versus iteration for base case assumptions starting from price-capped Cournot.

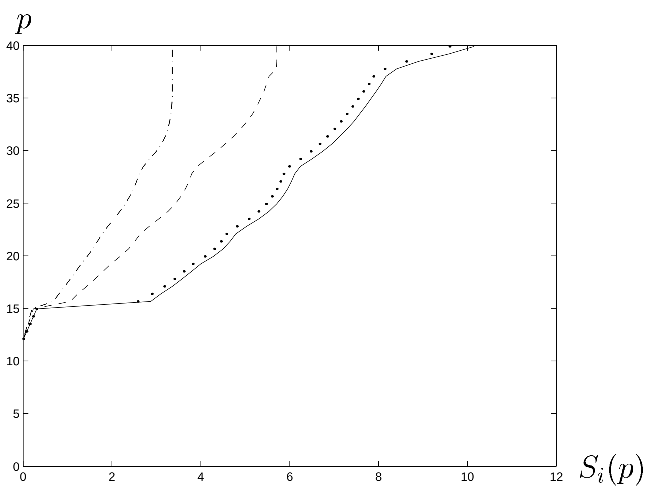

Figure 53: Supply functions at iteration 100 for base case assumptions starting from price-capped Cournot.

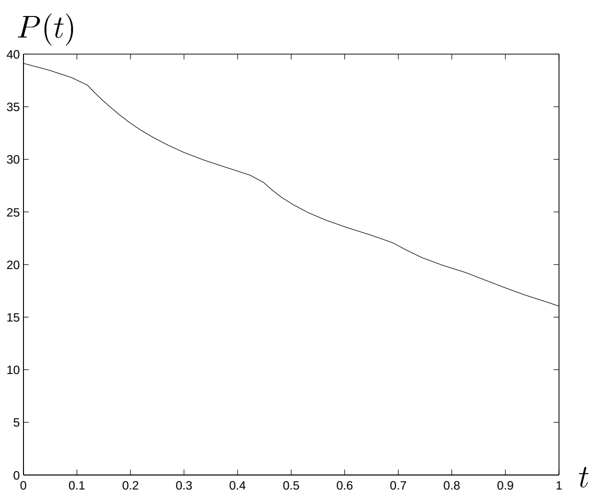

Figure 54: Price-duration curve at iteration 100 for base case assumptions, starting from price-capped Cournot.

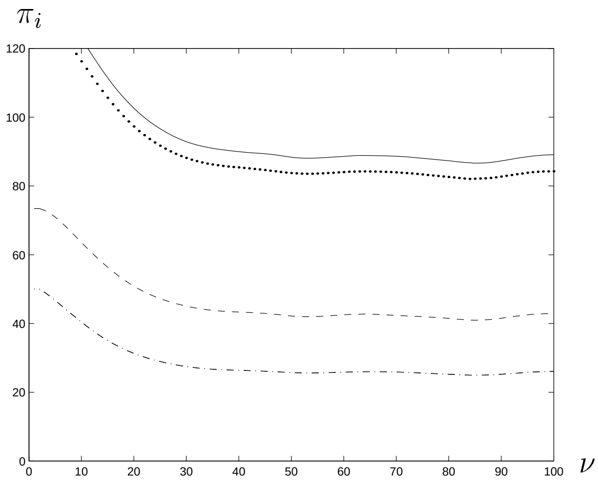

Figure 55: Profits versus iteration for base case assumptions starting from price-capped Cournot but with reduced price minimum of $\underline{p} = 8$ pounds per MWh.

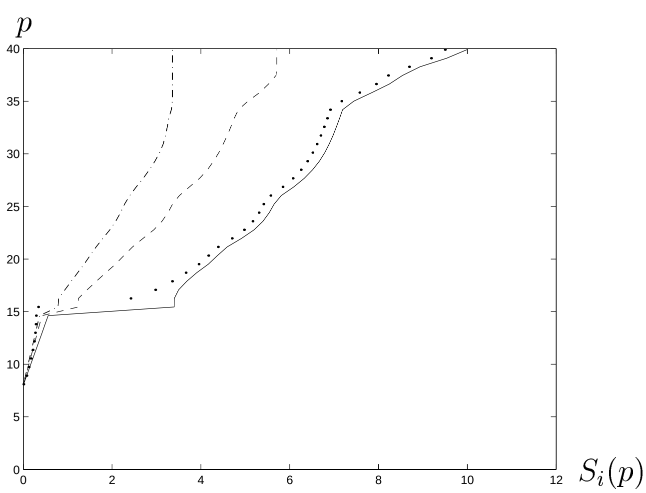

Figure 56: Supply functions at iteration 100 for base case assumptions starting from price-capped Cournot but with reduced price minimum of $\underline{p} = 8$ pounds per MWh.

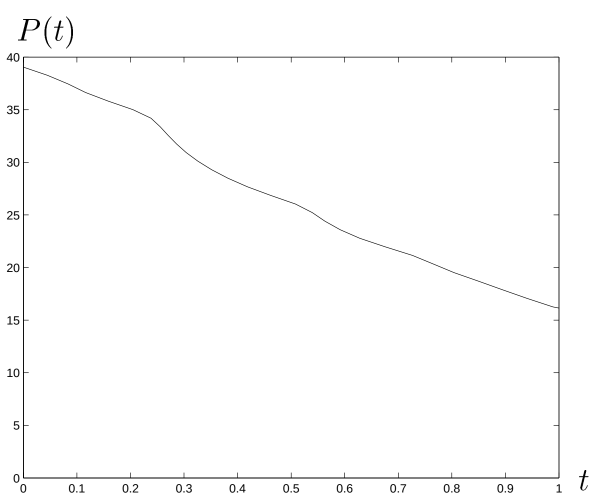

Figure 57: Price-duration curve at iteration 100 for base case assumptions, starting from price-capped Cournot but with reduced price minimum of $\underline{p} = 8$ pounds per MWh.

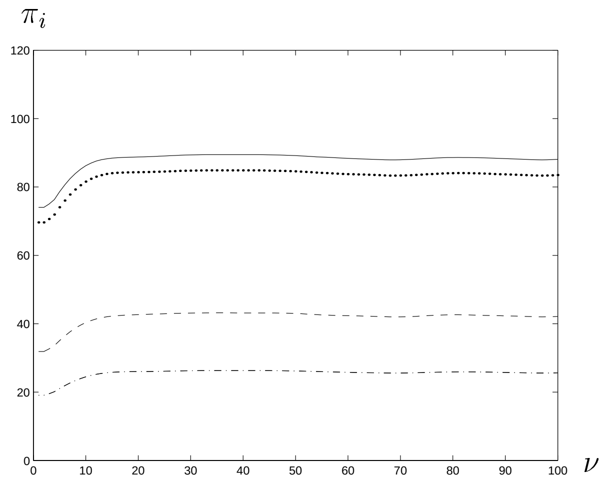

Figure 58: Profits versus iteration for base case assumptions starting from capacitated affine SFE, except that supply functions have 20 break-points.

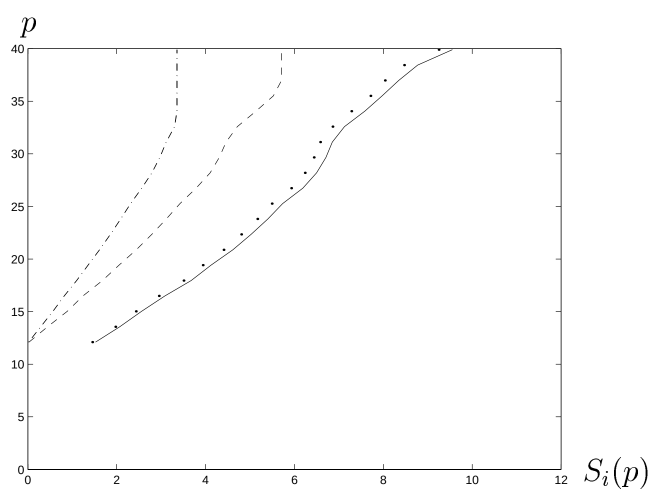

Figure 59: Supply functions at
iteration 100 for base case assumptions starting from capacitated affine SFE, except that supply functions have 20 break-points.

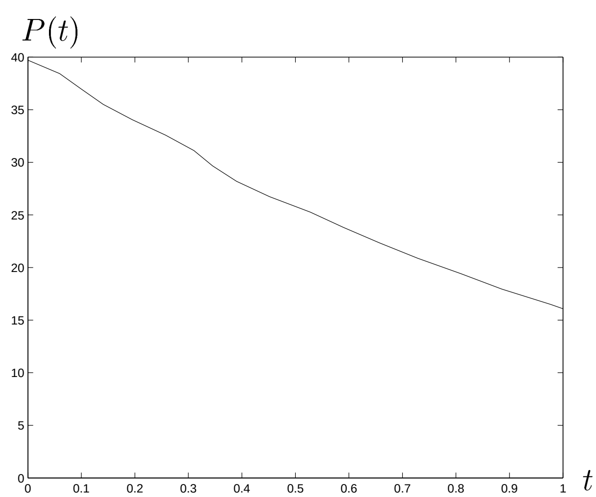

Figure 60: Price-duration curve at
iteration 100 for base case assumptions starting from capacitated affine SFE, except that supply functions have 20 break-points.

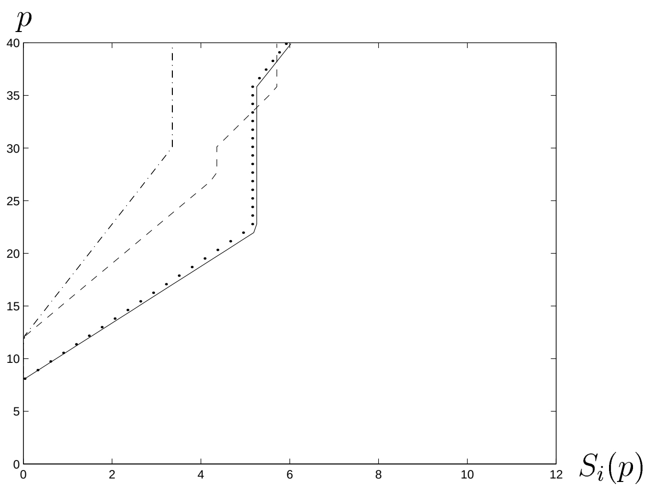

Figure 61: Supply function constructed according to recipe in [8].

11.3.6 Summary

Given the base case supply and demand configuration, the results at iteration 100 from all the starting functions, as shown in figures 46–60, are similar from the perspectives of:

- the profits,
- the general shape of the supply functions over the range of realized prices, (between about 16 and 40 pounds per MWh), and
- the form of the price-duration curve.

The supply functions at iteration 100 differ in detail in the range of realized prices depending on the starting function. For example, there are points of apparent non-differentiability in the supply functions and the location of these points differs from starting function to starting function. However, all starting functions have evidently converged towards similar equilibria. That is, for the base case there is only a very small range of multiple equilibria.

All firms have roughly the same marginal costs at peak capacity of approximately 27 pounds per MWh. However, in the supply functions at iteration 100, the largest two firms, 3 and 4, maximize their profits by withholding capacity so that prices are well in excess of 27 pounds per MWh for more than 45% of the time horizon.

The smallest two firms, 2 and 5, (represented by the leftmost of the supply function curves) bid in all their capacity when prices reach about 33 pounds per MWh. In contrast, the largest two firms do not provide all their capacity until the price reaches the price cap of 40 pounds per MWh. Moreover at prices above 35 pound per MWh, the largest two firms behave much less competitively than they do at low prices because the other three firms have reached their full capacity and no longer contribute to the slope of the residual demand faced by the large firms.

The supply functions of firms 1, 2, and 5 are concave over most of the range realized prices. These firms are at their capacity constraints at the peak realized capacity, so the concavity of their supply functions does not indicate an unstable equilibrium. On the other hand, firms 3 and 4 are not at their capacity constraints. Note that their supply functions at prices near to the maximum realized price are not concave. This is also consistent with the stability analysis in section 5.

The price at peak demand is just below the price cap. The prices at lower demands are significantly lower. SFE competition combined with a price cap has prevented prices from staying near to the price cap, except at peak demand.

For prices below about 20 pounds per MWh, corresponding to the right hand third of the price-duration curve, the supply functions and the price-duration curve are similar in appearance to the uncapacitated case. (Compare, for example, to figures 31 and 32, respectively.) However, it is clear that the capacity constraints have caused a significant shift in the supply function for prices above 20 pounds per MWh even though production at this price is only less than half of capacity. The presence of capacity constraints causes significant price mark-ups even at demands far below the peak.

Despite the considerable mark-ups, the prices are considerably lower than if the firms were to bid the price-capped Cournot supply function. (Compare the prices to figure 26.) The requirement that the bids be consistent across the time horizon has significantly affected the outcome, reducing equilibrium profits to about half what they would be if the price-capped Cournot supply functions were bid.

Conversely, the prices are considerably higher for much of the time horizon than if each firm were to bid the capacitated affine SFE starting function. This confirms that it is important to explicitly consider the effect of the capacity constraints on the equilibrium and that the equilibrium supply functions are not well approximated by naively truncating an uncapacitated SFE solution.

In [8], an *ad hoc* approach is taken to incorporating capacity constraints. Applying the recipe in [8] for constructing supply functions results in figure 61. The recipe in [8] provides a reasonable estimate of the equilibrium supply bids in this case for firms 1,2, and 5 (the smallest three firms). However, the recipe predicts less supply than the calculated equilibria for firms 3 and 4 at high prices.

The recipe in [8] does not explicitly consider the load-duration characteristic. The recipe sets supply at high prices based only on competition between firms 3 and 4 at high prices, but the effect of this is to limit the supply of these generators at lower prices. (See the vertical part of the supply curves for firms 3 and 4 between about 22 and 37 pounds per MWh in figure 61.) The recipe fails to fully value the sales opportunities for firms 3 and 4 at prices between 22 and 37 pounds per MWh. In general, any recipe that seeks to define the supply function independently of the load-duration characteristic will fail to make the profit maximizing trade-off between withholding at high prices and sales opportunities at low prices.

11.4 Varying the price cap

In this section we consider varying the price cap.

11.4.1 Starting from price-capped Cournot

Figure 62 shows the profits versus iteration ν for the base case assumptions starting from the price-capped Cournot supply function, except that the price cap was increased to 50 pounds per MWh. Figure 63 shows the supply functions at iteration 100. The price-duration curve for iteration 100 is shown in figure 64. (Note that the price axes on these graphs has a different scale to the previous graphs.)

Compared to the base case, the supply functions for the increased price cap case are significantly different for prices above 30 pounds per MWh. That is, the price cap affects supply at prices well below the price cap. In particular, raising the price cap yields further withholding of supply compared to the base case even at prices well below the base case price cap. Profits are up to 20% higher than in the base case, due primarily to the withholding of supply until prices become close to the price cap. This suggests that there is considerable value in being able to estimate the maximum marginal cost of generation to set a fairly tight price cap.

The observation that price caps deter the exercise of market power is well-known from single period models of interaction [16]. In the SFE case, a further issue is that the price stays well below the price cap at off-peak times. That is, the imposition of a single price cap applying at all times together with the requirement that supply functions remain fixed over an extended horizon has a similar effect to price caps that vary with demand conditions.

11.4.2 Price cap and multiple equilibria

Figure 65 shows the peak realized price at iteration 100 versus the price cap for price caps in the range of 30 pounds per MWh to 80 pounds per MWh. For each price cap, the result at iteration 100 for the price-capped Cournot starting function is shown as a cross while the result at iteration 100 for the capacitated competitive starting function is shown as a circle.

For price caps below about 40 pounds per MWh the peak realized price comes within about 1 pound per MWh of the price cap. The firms can coordinate on achieving close to the price cap. Moreover, for a given price cap the results at iteration 100 are very similar for both the price-capped Cournot and capacitated competitive starting functions. (The profits and supply functions at iteration 100 are also similar for the price-capped Cournot and the capacitated competitive starting functions for each value of the price cap below about 40 pounds per MWh.) That is, when the price cap is binding, there appears to be only a small range of equilibria exhibited.

In contrast, for values of the price cap above about 50 pounds per MWh the peak realized price at iteration 100 is in the range of around 45–50 pounds per MWh and there are non-trivial differences between the peak realized prices at iteration 100 for the price-capped Cournot and the capacitated competitive starting functions. (However, in some of the cases, the profit functions were still changing by more than 0.1% at each iteration, so some of the difference between the Cournot and competitive starting functions may be because the results at iteration 100 are not close enough to equilibrium.) This suggests that when the price cap is not binding there is a range of exhibited equilibria.

As discussed in section 6.2.1, we can calculate competitive and Cournot outcomes for the peak demand conditions. The price at peak demand for competitive bids is $p_0^{\text{comp}} \approx 27$

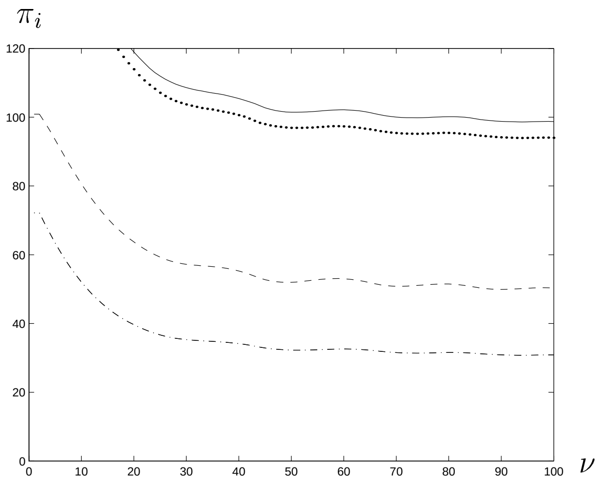

Figure 62: Profits versus iteration for base case assumptions starting from price-capped Cournot, except for increased price cap.

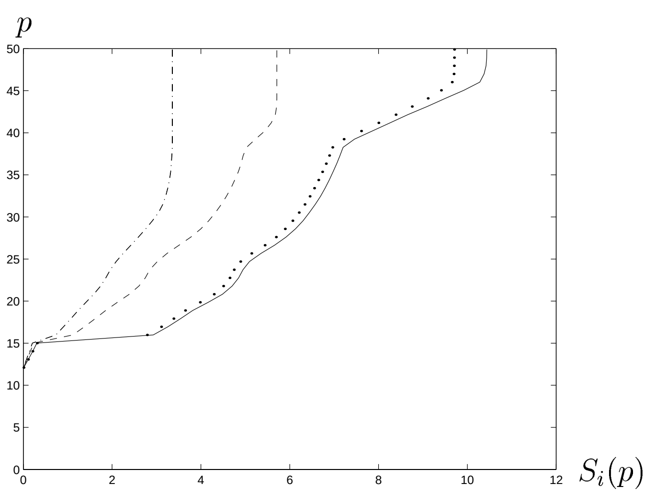

Figure 63: Supply functions at iteration 100 for base case assumptions starting from price-capped Cournot, except for increased price cap. (Note that that price axis is scaled differently compared to previous figures.)

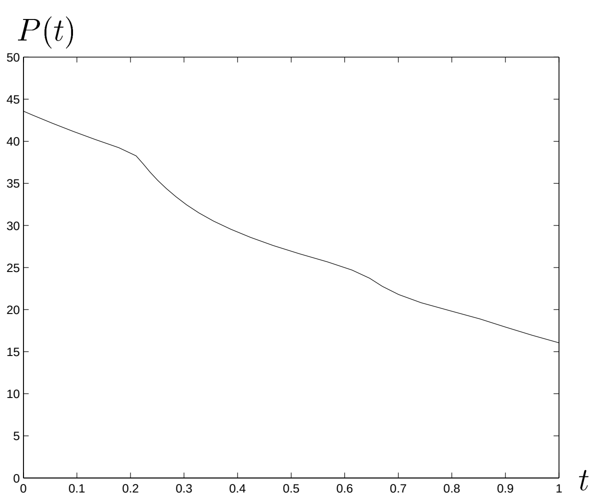

Figure 64: Price-duration curve at iteration 100 for base case assumptions starting from price-capped Cournot, except for increased price cap. (Note that the price axis is scaled differently compared to previous figures.)

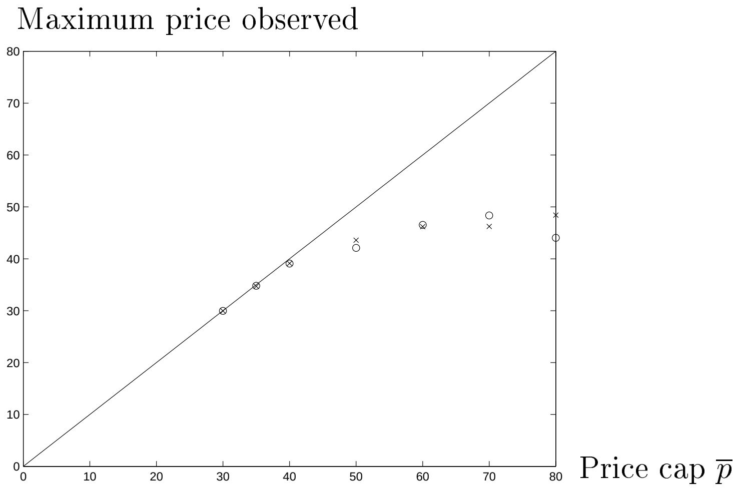

Figure 65: Maximum observed price versus price cap. Results starting from price-capped Cournot are shown with crosses, while results starting from capacitated competitive are shown with circles.

pounds per MWh. The price at peak demand under Cournot competition is $p_0^{\text{Cournot}} \approx 80$ pounds per MWh.

Even when the price cap is raised to 80 pounds per MWh, the peak realized price at iteration 100 for the supply function bids is far below 80 pounds per MWh for either starting function. The range of peak realized prices at iteration 100 for the price-capped Cournot and the capacitated competitive starting functions is relatively small compared to the peak Cournot price of 80 pounds per MWh.

In summary, when the price cap is binding on behavior, the range of exhibited equilibria seems to be very narrow. The price-capped Cournot and the capacitated competitive starting functions yield essentially the same results at iteration 100. When the price cap is not binding on behavior, there is a range of equilibrium outcomes; however, this range is relatively small compared to the difference between the price-capped Cournot and the capacitated competitive starting functions.

11.5 Increased capacities

We increased the capacity of all firms by 5% compared to the base case. The results for the price-capped Cournot starting function are shown in figures 66–68. The results for the competitive starting function are shown in figures 69–71. The profits at iteration 100 are approximately 20% lower for the capacitated competitive starting function compared to the price-capped Cournot starting function. The range of equilibrium profits is about 12% of the difference in profits between the price-capped Cournot and capacitated competitive supply functions.

In this case, firms 1, 2, and 5 reach their capacity below the peak realized price, but the price cap is not binding. There is a small range of equilibria in this case.

11.6 Increased load factor

The load duration characteristic in the base case has a relatively small load factor of around 30% implying that the supply functions were required to be set for a very long period or

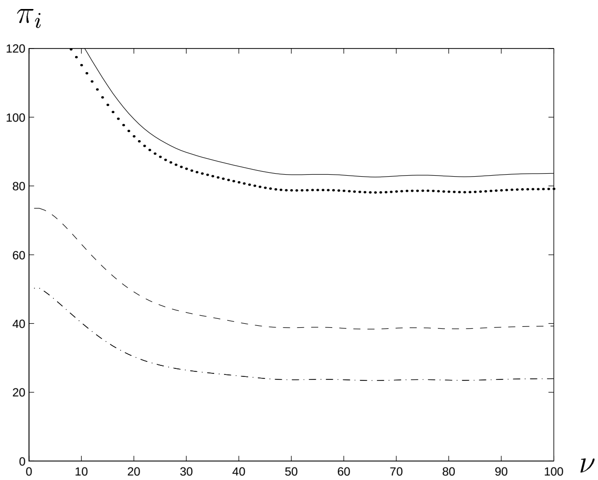

Figure 66: Profits versus iteration for base case assumptions except for 5% increase in all capacities, starting from price-capped Cournot.

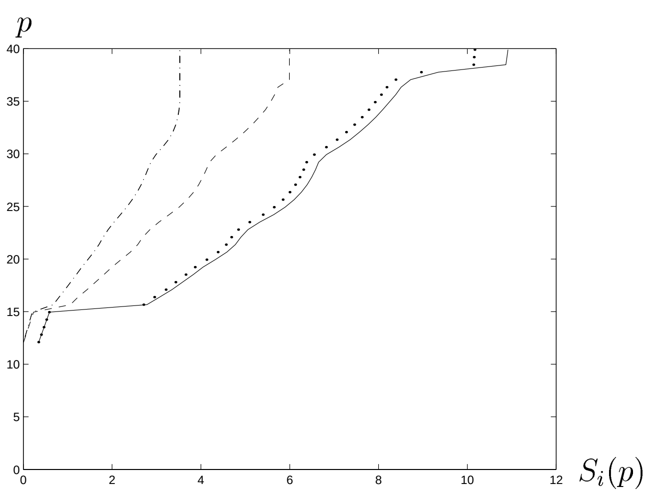

Figure 67: Supply functions at iteration 100 for base case assumptions except for 5% increase in all capacities, starting from price-capped Cournot.

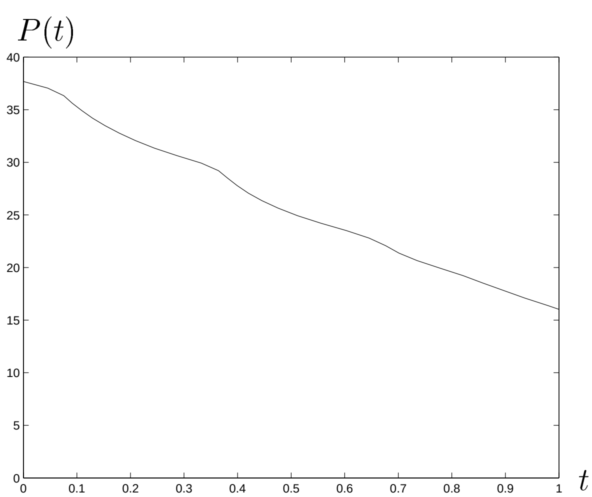

Figure 68: Price-duration curve at iteration 100 for base case assumptions except for 5% increase in all capacities, starting from price-capped Cournot.

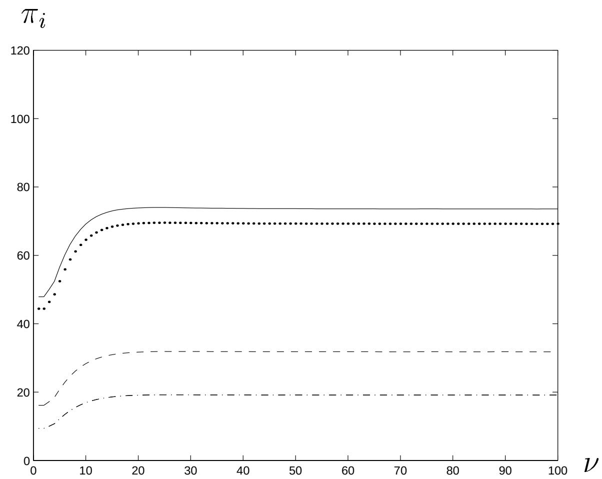

Figure 69: Profits versus iteration for base case assumptions except for 5% increase in all capacities, starting from capacitated competitive.

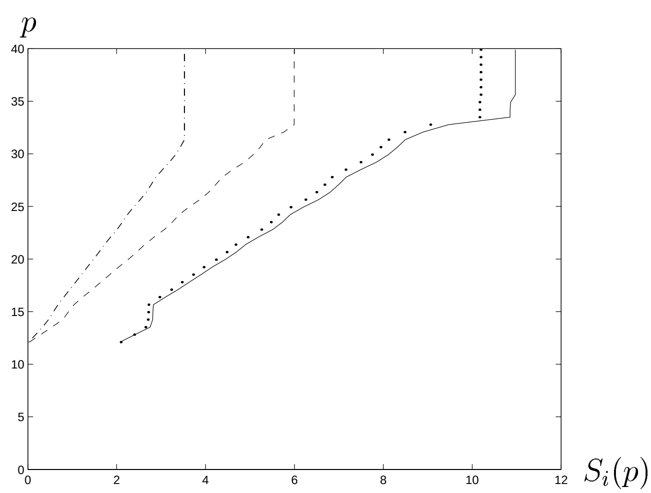

Figure 70: Supply functions at iteration 100 for base case assumptions except for 5% increase in all capacities, starting from capacitated competitive.

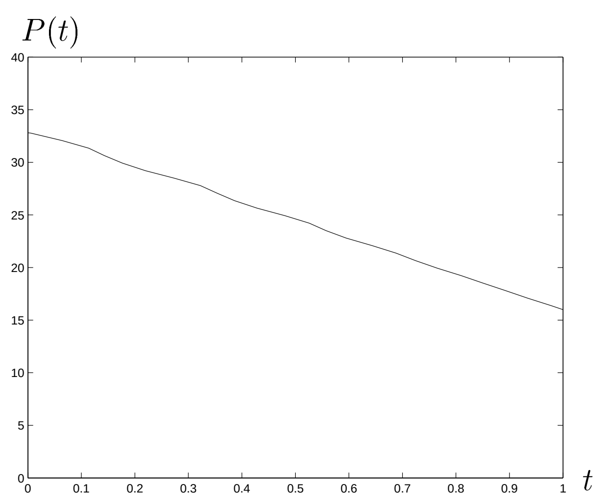

Figure 71: Price-duration curve at iteration 100 for base case assumptions except for 5% increase in all capacities, starting from capacitated competitive.

that a significant amount of demand was supplied by baseload capacity at prices at or below the price minimum $\underline{p}$ or that much of the demand was supplied under forward contracts. We divided the time horizon into peak and off-peak conditions and considered the case where bids were made separately for peak and off-peak conditions.

11.6.1 Peak conditions

We investigated a case where the load duration characteristic ranged linearly from 20 to 35 GW. This implies a load factor of around 60%. That is, we shortened the time horizon compared to the base case by omitting the off-peak times, but the time horizon still covered the peak conditions.

Figure 72 shows the profits versus iteration ν for the base case assumptions starting from the price-capped Cournot supply function, except that the load-duration characteristic has been changed so that $N(1) = 20$. (The value $N(0)$ was kept at 35.) As previously, firms 2 and 5 have identical costs and capacities, so they appear superimposed. The profit functions are not directly comparable to previous cases since the demand conditions have changed.

Figure 73 shows the supply functions at iteration 100. The supply functions at iteration 100 are very similar to the base case supply functions at iteration 100, over the range of realized prices. The price-duration curve for iteration 100 is shown in figure 74.

The results at iteration 100 for the capacitated competitive starting function are essentially the same as for the price-capped Cournot starting function. That is, it appears that the increase in the load factor has not significantly increased the range of equilibria.

11.6.2 Off-peak conditions

We also investigated a case where the load duration characteristic ranged linearly from 10 to 20 GW. That is, we shortened the time horizon compared to the base case by omitting the peak times. In this case, the price cap is not binding and so, as in the uncapacitated case and the increased capacity case, there are multiple solutions having a range of profits. The range of profits at iteration 100 is around 10% of the difference between the profits for the capacitated competitive and price-capped Cournot supply functions.

11.7 Increasing demand

Finally, we considered increase in demand with the same supply conditions as the base case. The demand was increased so that rationing was required.

11.7.1 Starting from capacitated affine SFE

Figure 75 shows the profits versus iteration ν for the base case assumptions starting from the capacitated affine SFE supply function, except that the load-duration characteristic has been changed so that $N(0) = 40$. (The value $N(1)$ was kept at 10.) In this case there is not enough capacity to meet demand at the peak. As previously, firms 2 and 5 have identical costs and capacities, so they appear superimposed.

Figure 76 shows the supply functions at iteration 100. The supply functions at iteration 100 are similar to the base case. That is, the difference in profits compared to the base case

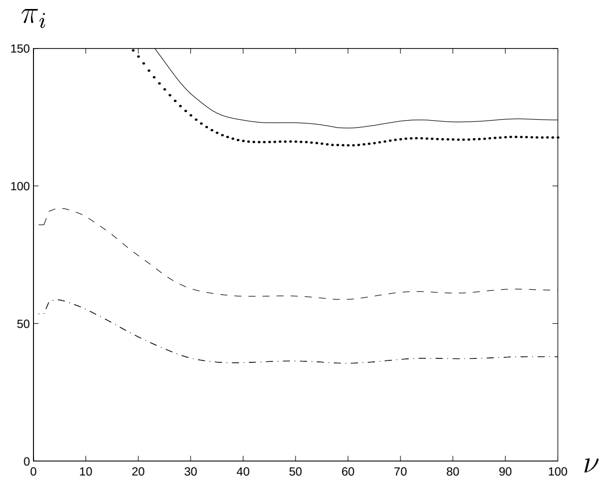

Figure 72: Profits versus iteration for base case assumptions, except for increased value of $N(1)$. (Note that the profit axis is scaled differently compared to previous figures.)

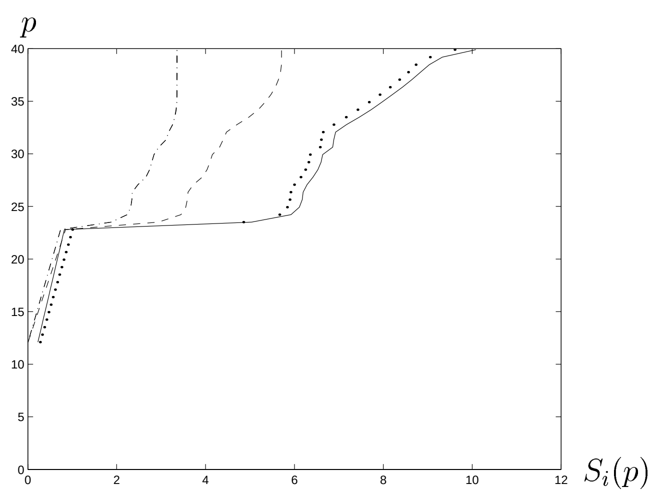

Figure 73: Supply functions at iteration 100 for base case assumptions, except for increased value of $N(1)$.

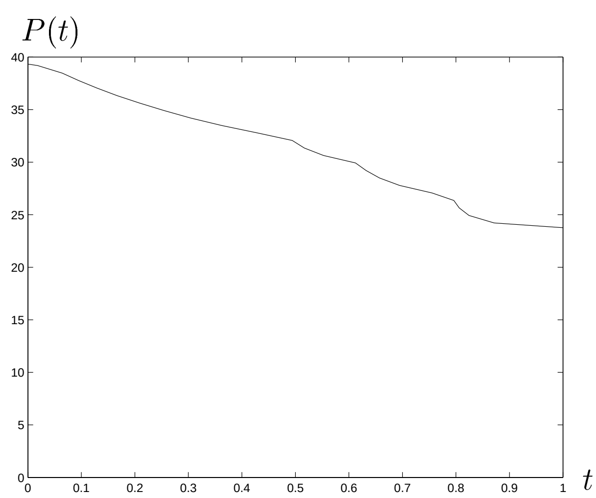

Figure 74: Price-duration curve at iteration 100 for base case assumptions, except for increased value of $N(1)$.

is primarily due to the higher demand in this case, rather than due to changed behavior because of tightened demand conditions. The price-duration curve for iteration 100 is shown in figure 77.

11.7.2 Starting from price-capped Cournot

Figure 78 shows the profits versus iteration ν for the base case assumptions starting from the price-capped Cournot supply function, except that $N(0) = 40$. Figure 79 shows the supply functions at iteration 100. The price-duration curve for iteration 100 is shown in figure 80. The results at iteration 100 are very similar to the case of starting from the capacitated affine SFE supply function.

11.7.3 Starting from capacitated affine SFE with high price cap

Figure 81 shows the profits versus iteration ν for the base case assumptions starting from the capacitated affine SFE supply function, except that $N(0) = 40$ and the price cap is set to $\overline{p} = 50$ pounds per MWh. Note that the profit axis has changed compared to previous figures because the profits are considerably higher. Figure 82 shows the supply functions at iteration 100. The price-duration curve for iteration 100 is shown in figure 83. Note that the price axes have been changed compared to some of the previous figures.

11.7.4 Bid caps

The previous cases were tested with the alternate rule of market wide bid caps instead of price caps. The bid supply functions were not significantly different in this case; however, profits were higher than for price caps because prices exceeded the bid cap whenever supply is tight.

11.7.5 Summary

Profits are considerably higher than in the previous cases. However, for the price cap of 40 pounds per MWh, most of the difference in profits compared to the base case is due to increased demand alone rather than changes in bid behavior. Despite the greater potential for exploitation of market power due to the need for rationing, the presence of the price cap and the requirement to bid consistently across the time horizon has limited the scope to increase profits.

In the case of the high price cap, however, the combination of the need for rationing and the increased price cap has led to even higher profits. The two firms with large capacity can withhold capacity until high prices are reached. This again demonstrates the value of a fairly tight price cap.

11.8 Characteristics of equilibrium solutions

As argued in corollary 11 of section 7, the solutions are always strictly increasing. As suggested in section 6.4, the equilibrium supply functions exhibit discontinuities in their

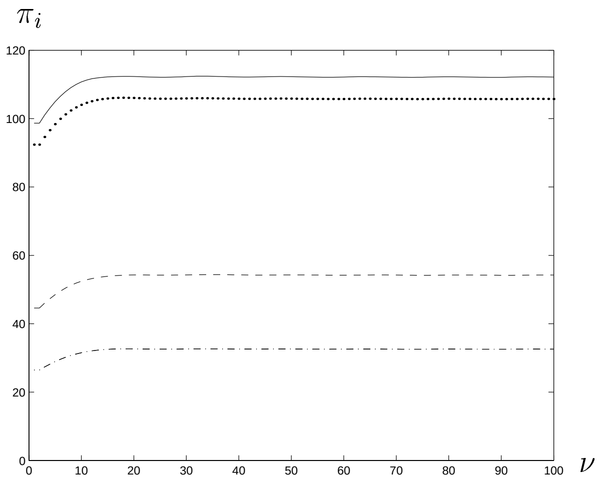

Figure 75: Profits versus iteration for case of rationing, starting from capacitated affine SFE.

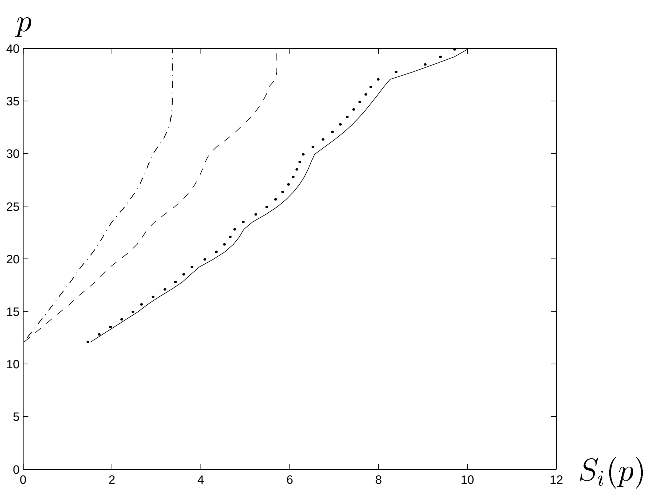

Figure 76: Supply functions at iteration 100 for case of rationing, starting from capacitated affine SFE.

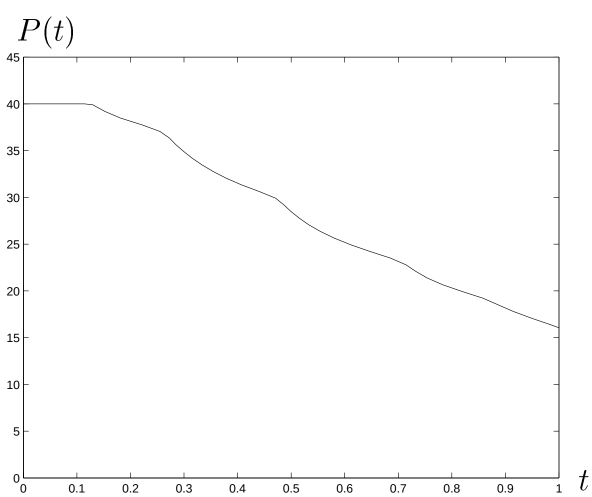

Figure 77: Price-duration curve at iteration 100 for case of rationing, starting from capacitated affine SFE.

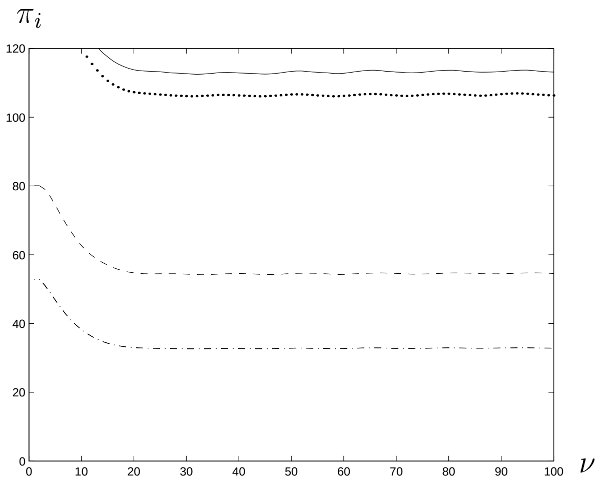

Figure 78: Profits versus iteration for case of rationing starting from price-capped Cournot.

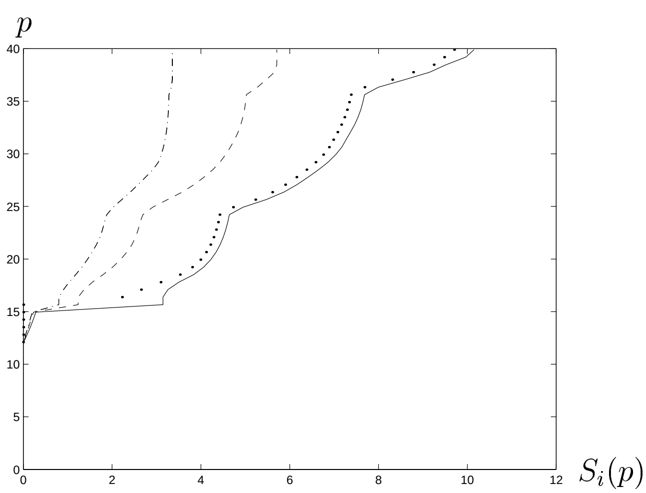

Figure 79: Supply functions at iteration 100 for case of rationing starting from price-capped Cournot.

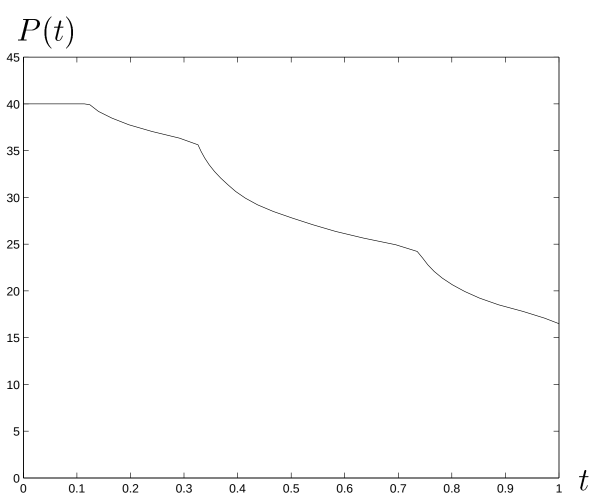

Figure 80: Price-duration curve at iteration 100 for case of rationing starting from price-capped Cournot. (Note that the price axis is scaled differently compared to previous figures.)

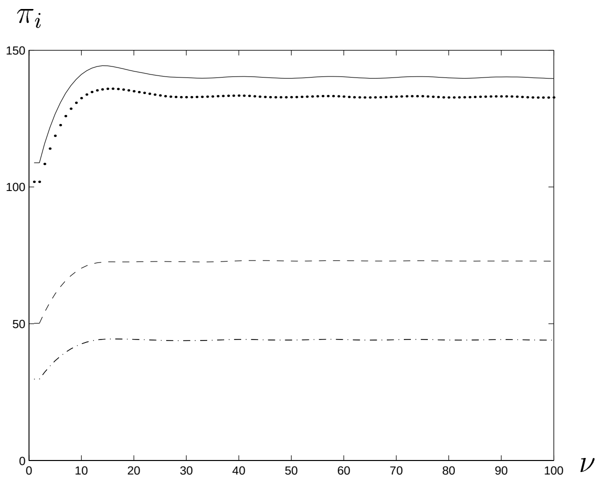

Figure 81: Profits versus iteration for high demand and high price cap. (Note that the profit axis is scaled differently compared to previous figures.)

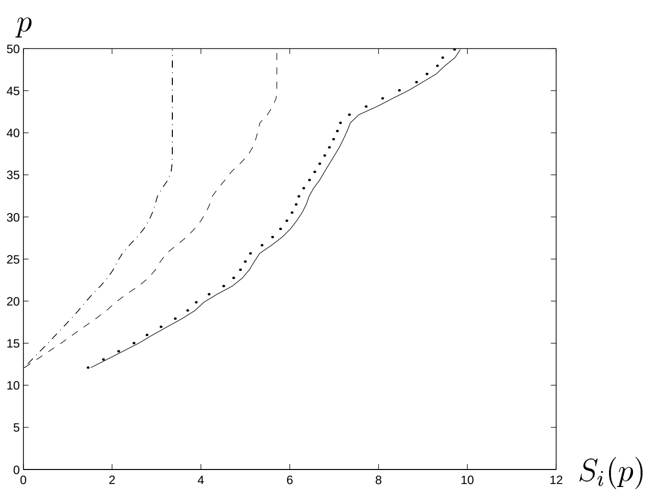

Figure 82: Supply functions at iteration 100 for high demand and high price cap. (Note that the price axis is scaled differently compared to previous figures.)

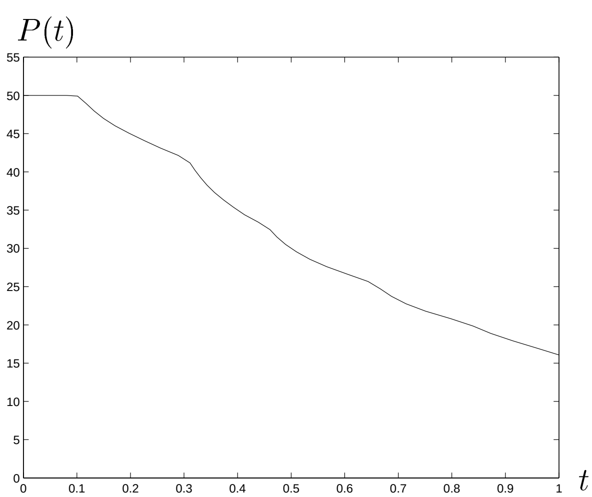

Figure 83: Price-duration curve at iteration 100 for high demand and high price cap. (Note that the price axis is scaled differently compared to previous figures.)

derivatives. As shown in corollary 12, between the discontinuities, the equilibrium solutions are consistent with solutions of (16) corresponding to a subset of the firms.

12 Conclusion

The main results of this paper are:

- In markets with heterogeneous firms and capacity constraints, the differential equation approach to finding the equilibrium supply function may not be effective by itself because the non-decreasing constraints, which couple decisions across the time horizon, are likely to be binding. An alternate approach, of iterating in the space of supply functions, is computationally intensive and has theoretical drawbacks of its own. However, based on the case studied, it appears to produce consistent and useful results.

- The range of supply function equilibria may be very small when capacity is fairly tight and there are binding price caps. This market condition is the most critical from a market power perspective. Even when price caps are not binding, the range of stable equilibria appears relatively small compared to the difference between the competitive and the Cournot outcomes. This strengthens the case for SFE analysis when market rules require consistent bids across a time horizon, particularly when capacity constraints and price caps are binding.

- Requiring supply functions to remain fixed over an extended time horizon appears to reduce the incentive to mark up prices compared to the Cournot outcome. SFEs that achieve profits that are close to Cournot profits are unstable and consequently should not be observed in the market.

- A single price cap imposed at all times may have significant effects both on- and off-peak.

As discussed in Borenstein [16], there are various problems facing wholesale electricity markets. Borenstein discusses the value of long-term contracting, real-time pricing, and price caps to a smoothly functioning electricity market. As well as the advantages cited in [16], long-term contracting can also reduce the effective load factor in the day-ahead market, which can rule out some of the least competitive equilibria. In this paper, the analysis of stability and the numerical studies suggest that requiring bid functions to be consistent over an extended time horizon having a large variation of demand may also be valuable in mitigating extreme prices and market power.

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